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A class of topological actions

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Abstract: We review definitions of generalized parallel transports in terms of Cheeger-Simons differential characters. Integration formulae are given in terms of Deligne-Beilinson cohomology classes. These representations of parallel transport can be extended to situations involving distributions as is appropriate in the context of quantized fields.

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Contents

1. Introduction

Parallel transports and generalizations thereof have been repeatedly met both in mathematics [1 − 7] and in global aspects of gauge theories [8 − 10], which played a major role in elementary particle physics.

It has taken some time for the existing mathematics [11 − 15, 7] to become known to physicists [9, 10, 16 − 19].

At the semi-classical level one is lead to integrate objects more general than differential forms over cycles with a result defined modulo integers; Cheeger-Simons differential characters [2] are privileged candidates. Their integral representations in terms of Deligne-Beilinson smooth cohomology classes are particularly well adapted to field theory for two reasons : first of all, they involve locally defined fields subject to some gluing properties. Besides, they allow for natural generalizations well adapted to, at least, semi-classical quantization. Indeed the latter already requires regularizing (thickening) the integration cycles, an operation which can be performed easily within the Deligne-Beilinson cohomology framework. This operation is less naive than one might think; indeed the corresponding currents are not any longer differential forms (de Rahm currents) but Deligne-Beilinson classes. In view of this phenomenon we shall proceed in detail from the semi-classical situation for which the use of Cheeger-Simons characters is well adapted. In this case there exist canonical integral representations in terms of differential forms with discontinuous coefficients and therefore inappropriate for applications to quantum fields, even in the semi-classical approximation. Fortunately, these integral representations can be replaced by others with smooth coefficients. The latter are easily generalizable to situations involving distributions and therefore well adapted to quantum fields. There is however a price to pay: the differential forms involved in the classical formulae have to be replaced (non canonically) by Deligne-Beilinson smooth classes.

We start in section 2 with the prototype example of Maxwell's electromagnetism in which a functional integral is defined under "reasonable" hypotheses concerning the interaction with an external current. The rest of the paper is devoted to a sequence of constructions which give a mathematical foundation of the above hypotheses.

Section 3 proposes three equivalent ways to describe Cheeger-Simons differential characters in terms of the integration of Deligne-Beilinson cohomology classes.

Section 4 presents the natural generalizations required upon quantization: the integration of Deligne-Beilinson classes with distributional coefficients.

Section 5 contains our concluding remarks.

A number of technical details are collected in three appendices.

2. Maxwell Semi-Classical theory (à la Feynman)

While in a classical theory the action (when it exists) is optional (in principle, the equations of motion are sufficient), it becomes the keystone of the Feynman semi-classical point of view. Hence, such an action must be carefully defined. In the context of Maxwell's electromagnetism, we consider the euclidean action defined on a 4-dimensional, riemmannian,[1] compact manifold M_4

$$S_{EM} = \frac{1}{2} \int_{M_4} F \wedge *F \, + \, i \cdot \text{“} \int_{M_4} j \wedge A \text{”} \, . \tag{2.1}$$

Quotes emphasize that we have to make precise the meaning of the second integral since A is **not** a 1-form on M_4, but rather a connection on a U(1)-bundle over M_4, with curvature F. We defer until the next section a mathematically sound definition [2] of " $\int_{M_4} j \wedge A$" for j a 3-form with integral periods.

At this point, we only need to know that " $\int_{M_4} j \wedge A$" will be defined modulo $2\pi\mathbb{Z}$ and will fulfill the following natural property : if $A = A_0 + \alpha$ (with A_0 a fixed U(1)-connection and α a generic 1-form), then

$$\text{“} \int_{M_4} j \wedge A \text{”} = \text{“} \int_{M_4} j \wedge A_0 \text{”} + \int_{M_4} j \wedge \alpha \, . \tag{2.2}$$

Gauge invariance requires $\int_{M_4} j \wedge (g^{-1}dg) \in 2\pi\mathbb{Z}$ which is less restrictive than the "classical" requirement $\int_{M_4} j \wedge (g^{-1}dg) = 0$, commonly assumed [20, 21] to hold at the quantum level.

Once the choice of definition of the action integral with the above property has been made, we can try to evaluate the state [3] $(\hbar = 1)$

$$\left\langle e^{- \, i \cdot \text{“} \int_{M_4} j \wedge A \text{”}} \right\rangle = \int \mathcal{D}A \, e^{-\frac{1}{2} \int_{M_4} F \wedge *F \, - \, i \cdot \text{“} \int_{M_4} j \wedge A \text{”}} \, , \tag{2.3}$$

where A is a U(1)-connection. First let

$$A = A_0 + \alpha \, , \tag{2.4}$$

with A_0 a background connection and α a globally defined 1-form. Then, denoting by $F_0 = dA_0$ the background curvature, we obtain

$$\begin{aligned} \left\langle e^{- \, i \cdot \text{“} \int_{M_4} j \wedge A \text{”}} \right\rangle &= e^{-\frac{1}{2} \int_{M_4} F_0 \wedge *F_0 \, - \, i \cdot \text{“} \int_{M_4} j \wedge A_0 \text{”}} \times \\ &\times \int \mathcal{D}\alpha \, e^{-\frac{1}{2} \int_{M_4} d\alpha \wedge * d\alpha \, - \, \int_{M_4} F_0 \wedge * d\alpha \, - \, i \cdot \int_{M_4} j \wedge \alpha} \, . \end{aligned} \tag{2.5}$$

The 1-form α is linearly coupled to $(j + i \, d * F_0)$ and we need to gauge fix the α integration. Note that $\int j \wedge \alpha$ is an ordinary integral. Gauge transformations connected with the identity are eliminated by choosing a Green function (ξ, the gauge parameter)

$$G_\xi \; = \; [\delta d + \xi d \delta]^{-1} \; , \qquad \xi > 0 \, , \tag{2.6}$$

in the subspace orthogonal to harmonic forms (the elimination of large gauge transformations will come later). So, we are led to

$$\begin{aligned} \left\langle e^{- \, i \cdot \text{“} \int_{M_4} j \wedge A \text{”}} \right\rangle &= e^{-\frac{1}{2} \int_{M_4} F_0 \wedge *F_0 \, - \, i \cdot \text{“} \int_{M_4} j \wedge A_0 \text{”}} \times \\ &\times e^{-\frac{1}{2} \int_{M_4} (j + i \, d * F_0)_\perp G_\xi \, * (j + i \, d * F_0)_\perp} \times Z\left(j_\|\right) . \end{aligned} \tag{2.7}$$

The subscript $\perp$ (resp. $\|$) refers to the decomposition of forms into components orthogonal to (resp. along) harmonic forms. We shall come to the definition of $Z(j_\|)$ later.

---

[1] $*$ is the usual Hodge operator.

[2] It will turn out that more data than just the 3-form j will be needed.

[3] A linear functional on observables.

The A_0 dependence can be reduced to:

$$\left\langle e^{-\, i\cdot \text{“}\int_{M_4} j\wedge A\text{”}} \right\rangle = e^{-\frac{1}{2}\int_{M_4} F_{0\|}\wedge * F_0 \;-\; i\cdot \text{“}\int_{M_4} j_{\|}\wedge A_0\text{”}} \times$$
$$\times\, e^{-\frac{1}{2}\int_{M_4} j_\perp G_\xi * j_\perp} \cdot Z\left(j_{\|}\right) . \tag{2.8}$$

The first term yields an overall normalization factor to be divided out. The third term is ξ independent by $dj = 0$. The forms $\alpha_{\|}$ and $j_{\|}$ being harmonic are necessarily closed (also co-closed). Using Poincaré duality and assuming no torsion, we can decompose them along a dual basis of integral 3-cycles and 1-cycles respectively

$$\alpha_{\|} = \sum_k \alpha_k\, {\zeta_k}^{(3)} + d(\cdots) , \qquad j_{\|} = \sum_k n_k {\zeta_k}^{(1)} + d(\cdots)$$

with

$$\left\langle {\zeta_k}^{(3)} {\zeta_l}^{(1)} \right\rangle = \delta_{kl} , \tag{2.9}$$

where the α_k's are real numbers since α is real, while the n_k's are integers since $j_{\|}$ has integral periods. With this decomposition of $\alpha_{\|}$ and $j_{\|}$, we can formally write

$$Z(j_{\|}) = \int \mathcal{D}\alpha_{\|}\, e^{\,i\cdot\int_{M_4} \alpha_{\|}\wedge j_{\|}} = \int d\vec{\alpha}\, e^{\,i\vec{n}\cdot\vec{\alpha}} , \tag{2.10}$$

where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m)$ and $\vec{n} = (n_1, \ldots, n_m)$.

Now, large gauge transformations are :

$$\alpha_k \mapsto \alpha_k + p_k , \qquad p_k \in 2\pi\mathbb{Z} \tag{2.11}$$

and can be factored out by transforming α_k integration into ϑ_k integration $0 \leq \vartheta_k < 2\pi$:

$$Z(j_{\|}) = \int d\vec{\vartheta}\, e^{\,i\vec{n}\cdot\vec{\vartheta}} . \tag{2.12}$$

These angles ϑ_k parametrize $H^1(M_4, \mathbb{R})/H^1(M_4, \mathbb{Z})$, still assuming no torsion (torsion yields an extra factor).

Similarly

$$e^{\,i\cdot \text{“}\int_{M_4} j_{\|}\wedge A_0\text{”}} = e^{\,i\vec{n}\cdot\vec{\vartheta}_0} , \tag{2.13}$$

where ϑ_{0k} are fixed angles which may be incorporated into ϑ_k.

To conclude, after normalization, the state $\langle\ \rangle$ can be decomposed into gauge invariant states labelled by the angles $\vec{\vartheta}$

$$\left\langle e^{\,i\cdot \text{“}\int_{M_4} j\wedge A\text{”}} \right\rangle = \int d\vec{\vartheta} \left\langle e^{\,i\cdot \text{“}\int_{M_4} j\wedge A\text{”}} \right\rangle_{\vec{\vartheta}} , \tag{2.14}$$

with

$$\left\langle e^{\,i\cdot \text{“}\int_{M_4} j\wedge A\text{”}} \right\rangle_{\vec{\vartheta}} = e^{\,i\vec{n}\cdot\vec{\vartheta}} \cdot e^{-\frac{1}{2}\int_{M_4} j_\perp G_\xi * j_\perp} , \tag{2.15}$$

a familiar situation which provides an alternative to the commonly accepted choice [20, 21] which amounts to integrate over $\vec{\vartheta}$'s with the result $\propto \delta(j_{\|})$; in the latter case $j = dm$ are the only possible integration currents for A, while for the states defined in (2.15) the currents j are only required to be closed forms with integral periods. In other words, homological triviality of Wilson loops or appropriately smeared version thereof are not consequences of gauge invariance, but rather, of some form of locality.

3. Integral representations of differential characters

In section 2 we have described the physical consequences of " $\int_M j \wedge A$ " being defined modulo $2\pi\mathbb{Z}$ (with j a form with integral periods). We shall now proceed to give some substance to this assumption and write down explicit formulae.

To start with, let us recall that one can associate to any closed curve[4] Γ in a manifold M a closed current δ_Γ (i.e. a closed form whose local representatives have distributional coefficients) such that integration of a form ω along Γ formally reduces to the integration of $\delta_\Gamma \wedge \omega$ over the whole of M [22].

We shall first try to find a satisfactory definition of the *circulation integral* of A along a closed curve Γ by considering various situations. This study will naturally lead us to the mathematical notion of *differential character* introduced by Cheeger and Simons [2].

Then, while seeking for a representation of a differential character supported by Čech-de Rham cohomology theory, there will emerge a defining formula for " $\int_M j \wedge A$ " in terms of Deligne-Beilinson cohomology [13]. We will see that for $j = \delta_\Gamma$, there is a canonical definition of this integral, whereas for general j there is a whole class of adequate definitions.

From now on, M will be a **torsion-free** smooth n-dimensional oriented compact manifold without boundary.

3.1 Circulation of U(1) gauge fields as differential characters

Within Maxwell's theory of electromagnetism on M_4, due to the triviality of the homology and cohomology groups of $\mathbb{R}^4$ (i.e. any closed curve is a boundary, and any closed 3-form is exact), the circulation of a U(1)-gauge field A along a closed curve Γ is a perfectly well-defined and gauge invariant integral which measures the magnetic flux through any surface Σ with boundary $\Gamma = \partial\Sigma$, namely

$$\text{“}\oint_\Gamma A\text{”} \equiv \oint_{\Gamma=\partial\Sigma} A = \int_\Sigma F \,. \tag{3.1}$$

Of course, such a property fails for a general manifold M with non-trivial (co-)homology groups. Nevertheless, it may be asked whether (3.1) can be maintained for boundaries $\Gamma = \partial\Sigma$, assuming that "$\oint_\Gamma A$" has a mathematical meaning for any closed curve Γ in M. Let us then consider a closed curve Γ that splits a **closed** surface Σ into two components Σ_+ and Σ_- : $\Sigma = \Sigma_+ \cup \Sigma_-$ and $\Gamma = \partial\Sigma_+ = -\partial\Sigma_-$, where the minus sign takes care of orientations. Then, we would have

$$\text{“}\oint_\Gamma A\text{”} = \int_{\Sigma_+} F \tag{3.2}$$

since $\Gamma = \partial\Sigma_+$, and

$$\text{“}\oint_\Gamma A\text{”} = -\int_{\Sigma_-} F \tag{3.3}$$

since $\Gamma = \partial\Sigma_-$. Since F is a U(1) curvature, we know that

$$\oint_{\Sigma_-} F + \oint_{\Sigma_+} F = \oint_\Sigma F \in \mathbb{Z}(1) := 2i\pi\mathbb{Z} \tag{3.4}$$

[4] By *curve* we mean a 1-dimensional embedded smooth submanifold of M.

JHEP08(2005)027

on any closed surface Σ. This suggests that, if it exists, "$\oint_\Gamma A$" is only defined modulo $\mathbb{Z}(1) := 2i\pi\mathbb{Z}$. Otherwise stated, we can expect, for fixed A, "$\frac{1}{2i\pi}\oint_\Gamma A$" to be some $\mathbb{R}/\mathbb{Z}$-valued linear functional on the space of closed curves (cycles). Let us have a closer look at such an assumption.

To begin with, a U(1)-gauge transformation, g, changes the connection A into the connection $A^g = A + g^{-1}dg$ with the same curvature F; therefore, **if (3.1) holds**

$$\text{“}\oint_{\partial\Sigma} A^g\text{”} = \text{“}\oint_{\partial\Sigma} A + g^{-1}dg\text{”} = \int_\Sigma F = \text{“}\oint_{\partial\Sigma} A\text{”}, \tag{3.5}$$

i.e. "$\oint_{\partial\Sigma} A$" is gauge invariant.

In fact, for any closed 1-form α on M, $A + \alpha$ is also a connection with curvature $F = dA$, so that we obtain a relation similar to (3.5) with α in place of $g^{-1}dg$. Consequently, we can infer that connections with the same curvature may define (*a priori* different) $\mathbb{R}/\mathbb{Z}$-valued linear functionals on cycles which *coincide on boundaries*. In this sense, the "integral" of A on boundaries is completely defined by F.

For a general closed curve Γ and any gauge transformation g, we would like to maintain gauge invariance of "$\oint_\Gamma A$", which is not immediate since the term $\oint_\Gamma g^{-1}dg$ may not vanish (Γ not being necessarily a boundary). However, since $g^{-1}dg$ is the pullback by g of the standard U(1) ($\simeq S^1$) volume 1-form, $z^{-1}dz$, we have

$$\oint_\Gamma g^{-1}dg \in \mathbb{Z}(1)\,.$$

Accordingly, still assuming that "$\oint_\Gamma A$" is defined modulo $\mathbb{Z}(1)$, we obtain the sought after gauge invariance

$$\text{“}\oint_\Gamma A^g\text{”} = \text{“}\oint_\Gamma A\text{”}\,, \tag{3.6}$$

though Γ is **not** a boundary.

All these requirements can be satisfied if we ask for (3.1) and define "$\frac{1}{2i\pi}\oint_\Gamma A$" to be an $\mathbb{R}/\mathbb{Z}$-valued functional, linear in Γ and affine in A,[5], a property which is satisfied if we set

$$\text{“}\oint_\Gamma (A+\gamma)\text{”} = \text{“}\oint_\Gamma A\text{”} + \oint_\Gamma \gamma\,, \tag{3.7}$$

where the last integral is the ordinary integral of the 1-form γ -in the same line of thought recall (2.2)-. Then, for any closed 1-form α we have

$$\text{“}\oint_\Gamma (A+\alpha)\text{”} = \text{“}\oint_\Gamma A\text{”} \tag{3.8}$$

if and only if all periods of α take values in $\mathbb{Z}(1)$. In fact the 1-forms $g^{-1}dg$, with g running through the U(1)-gauge group, generate the space of closed 1-forms with $\mathbb{Z}(1)$-valued periods. That is, if $per(\alpha) \in \mathbb{Z}(1)$, we can write

$$\alpha = g^{-1}dg + d\lambda$$

[5]This is a natural demand since the space of connections is an affine space.

JHEP08(2005)027

for some U(1)-gauge transformation g and some function λ on M. Then, as far as "integration" of A on closed curves is concerned, gauge invariance is equivalent to invariance under $A \mapsto A + \alpha$, with α a form with $\mathbb{Z}(1)$-valued periods. Therefore, it is expected that two connections that differ by a form with $\mathbb{Z}(1)$-valued periods define the same $\mathbb{R}/\mathbb{Z}$-valued linear functional on the space of closed curves.

At this point, let us make some remarks. First, if the connection A is a 1-form on M (for instance when the corresponding U(1)-bundle is flat), we must require that the general definition of "$\oint_\Gamma A$" reduces to the usual definition of the integral of a form. Second, up to now, we have only considered U(1)-connections on M. In a more general situation we will consider objects $A^{(p)}$, representing antisymmetric tensor "gauge potentials" which appear in supergravities and string theories [23]. However, the geometric situation turns out to be more involved than in the case of connections. Indeed, a U(1)-connection, although it is not a 1-form on M, is lifted as a 1-form on some principal U(1)-bundle over M. Such $A^{(p)}$'s will in general not be p-forms on M. It turns out that they can be considered as connections on new mathematical objects called gerbes [13, 24]. Here we will not go into such an interpretation: we will consider locally defined differential forms "$A^{(p)}$" on M whose differentials, $F^{(p+1)}$, are globally defined $(p+1)$-forms with $\mathbb{Z}$-valued periods on M.[6] We will define an $\mathbb{R}/\mathbb{Z}$-valued linear functional,"$\oint_{S_p} A^{(p)}$" on the space of closed p-submanifolds, S_p, of M. Such linear functionals turn out to be *differential characters* in the sense of J. Cheeger and J. Simons. Differential characters have been constructed within the framework of Chern-Simons' theory of secondary characteristic classes, an extension of the Chern-Weil theory. They were introduced to describe, on the base space, secondary characteristic classes of principal bundles initially defined as differential forms on the whole bundle space (see [3] for a review, and [2] for the original reference).

Our integrals, "$\oint_{S_p} A^{(p)}$", are related to *Deligne-Beilinson cohomology classes* as presented in [13] and therefore (cf. section A.7) offer a parametrization of differential characters.

In appendix A the reader will find notations, basic definitions and results concerning smooth Deligne-Beilinson cohomology groups $H^q(\mathcal{C}_p, D)$.

Our basic example deals with a U(1)-connection on the n- dimensional manifold M. In this case there is a one to one correspondence between the second smooth Deligne cohomology group of M, $H^2(\mathcal{C}_2, D)$,[7] and the set of equivalence classes of U(1) principal bundles with connection, $(P[\mathrm{U}(1)], A)$ (cf. appendix C). We will show how to integrate an element of $H^2(\mathcal{C}_2, D)$ over a 1-cycle, z_1, and take this "integral" as a definition for "$\frac{1}{2i\pi}\oint_\Gamma A$". This generalizes to integrating elements of $H^{p+1}(\mathcal{C}_{p+1}, D)$ over p-cycles, z_p which provides a definition for "$\oint_{z_p} A^{(p)}$". As we shall see (section 3.2) the classical Weil construction, pertaining to singular homology, both suggests a natural definition of elements of $H^{p+1}(\mathcal{C}_{p+1}, D)$ and of their integration over a p-cycle. In [18] R. Zucchini gives integral representations of "relative" differential characters, essentially identical with ours, independently of the ex-

[6] In this framework, $A = (2i\pi)A^{(1)}$, and its curvature $F = (2i\pi)F^{(2)}$.

[7] cf. appendix B.

pression of the integrand in terms of Deligne-Beilinson classes. Later (section 3.3) we will give another definition of the integral which avoids Weil's analysis of the cycle and allows for generalization.

3.2 Integration over a cycle: the appearance of Deligne-Beilinson classes

There is a natural procedure to define integration over integral cycles, based on the classic work of André Weil [25]. In this paper, for any simple[8] covering $\mathcal{U}$ of M, "$\mathcal{U}$-p-chains" are defined as singular p-chains, C_p, such that

$$C_p = \partial C_{(0,p)} := \sum_{\alpha} C_{(0,p),\,\alpha}\,, \tag{3.9}$$

where every $C_{(0,p),\,\alpha}$ is a singular p-chain with carrier $\mathcal{U}_\alpha$ (here, ∂ is the boundary operator on Čech chains). A $\mathcal{U}$-p-cycle z_p is a closed $\mathcal{U}$-p-chain ($bz_p = 0$, with b the boundary operator on singular chains). Then, it is shown that for any $\mathcal{U}$-p-cycle z_p of M there exists a sequence of Čech (smooth) singular $\mathcal{U}$-chains, $z_{(k,p-k)}$

$$z^{\mathcal{W}}_{(p)} := \left(z_{(0,p)}, \ldots, z_{(k,p-k)}, \ldots, z_{(p,0)}\right)\,, \tag{3.10}$$

where each $z_{(k,p-k)}$ has support in some open $(k+1)$-fold intersection of $\mathcal{U}$, such that

$$\begin{aligned}
\partial z_{(0,p)} &= z_{(-1,p)} := z_p \\
b z_{(k,p-k)} &= \partial z_{(k+1,p-k-1)}\,, \qquad k \in \{1, \ldots, p-1\} \\
b_0 z_{(p,0)} &:= z_{(p,-1)}\,,
\end{aligned} \tag{3.11}$$

where b_0 is just the "degree" operator on singular chains [25], $z_{(p,-1)}$ is an integral Čech p-cycle of $\mathcal{U}$ and $(\partial z_{(k,p-k)})_{\alpha_0\cdots\alpha_{k-1}} = \sum_\beta z_{(k,p-k),\beta\alpha_0\cdots\alpha_{k-1}}$.

The collection $z^{\mathcal{W}}_{(p)}$ is called a Weil descent of z_p, and the corresponding equations (3.11) a Weil descent equation of $z^{\mathcal{W}}_{(p)}$.

Now, if $Z^{\mathcal{W}}_{(p)}$ is another Weil descent of the same $\mathcal{U}$-p-cycle z_p, it differs from $z^{\mathcal{W}}_{(p)}$ according to

$$\begin{aligned}
Z_{(0,p)} &= z_{(0,p)} + \partial t_{(1,p)} + b t_{(0,p+1)}\,, \\
Z_{(k,p-k)} &= z_{(k,p-k)} + b t_{(k,p-k+1)} + \partial t_{(k+1,p-k)}\,, \qquad k = 1, \ldots, p-1 \\
Z_{(p,0)} &= z_{(p,0)} + b t_{(p,1)} + \partial t_{(p+1,0)}\,,
\end{aligned} \tag{3.12}$$

where the $t_{(k,p-k+1)}$ are some Čech $\mathcal{U}$-chains. Since z_p is fixed, we must have

$$\partial b\, t_{(0,p+1)} = 0 = b \partial\, t_{(0,p+1)}\,, \tag{3.13}$$

which means that $\partial t_{(0,p+1)}$ is a $\mathcal{U}$-$(p+1)$-cycle, $\tilde{z}_{p+1}$ which in turn gives rise to a Weil descent

$$\tilde{z}^{\mathcal{W}}_{(p+1)} := \left(\tilde{z}_{(0,p+1)} := t_{(0,p+1)}, \tilde{z}_{(1,p)}, \ldots, \tilde{z}_{(k,p-k+1)}, \ldots, \tilde{z}_{(p+1,0)}\right)\,,$$

[8] Definitions and notations are given in appendix A.

so that

$$
\begin{aligned}
Z_{(0,p)} &= z_{(0,p)} + \partial\big(t_{(1,p)} + \tilde{z}_{(1,p)}\big)\,, \\
Z_{(k,p-k)} &= z_{(k,p-k)} + b\big(t_{(k,p-k+1)} + \tilde{z}_{(k,p-k+1)}\big) + \partial\big(t_{(k+1,p-k)} + \tilde{z}_{(k+1,p-k)}\big)\,, \\
Z_{(p,0)} &= z_{(p,0)} + b\big(t_{(p,1)} + \tilde{z}_{(p,1)}\big) + \partial\big(t_{(p+1,0)} + \tilde{z}_{(p+1,0)}\big)\,,
\end{aligned} \tag{3.14}
$$

with $k = 1, \ldots, p-1$. Accordingly, the general ambiguities on a Weil descent of a given cycle z_p of M take the form

$$
\begin{aligned}
Z_{(0,p)} &= z_{(0,p)} + \partial h_{(1,p)}\,, \\
Z_{(k,p-k)} &= z_{(k,p-k)} + b h_{(k,p-k+1)} + \partial h_{(k+1,p-k)}\,, \\
Z_{(p,0)} &= z_{(p,0)} + b h_{(p,1)} + \partial h_{(p+1,0)}\,,
\end{aligned} \tag{3.15}
$$

By identifying Weil descents that differ by ambiguities (3.15), one defines an equivalence relation between Weil descents whose corresponding equivalence classes *canonically* represent $\mathcal{U}$-p-cycles of M. Actually, one could introduce a boundary operator made of the operators b and ∂, turning what we have just done into a homological game in which Weil descent classes are homology classes.

Similarly — cf. appendix A —, a sequence

$$
\omega_{\mathcal{D}}^{(p)} := \left(\omega^{(0,p)}, \omega^{(1,p-1)}, \ldots, \omega^{(p,0)}, \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}\right), \tag{3.16}
$$

where $\omega^{(k,p-k)} \in \check{\mathrm{C}}^{k}(\mathcal{U}, \Omega^{(p-k)}(M))$ and $\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \in \check{\mathrm{C}}^{(p+1)}(\mathcal{U})$ defines a Deligne-Beilinson cocycle if

$$
\left(\tilde{d} + \delta\right)\omega_{\mathcal{D}}^{(p)} = D\omega_{\mathcal{D}}^{(p)} = 0\,,
$$

i.e.

$$
d_{p-k}\omega^{(k,p-k)} = \delta\omega^{(k-1,p-k+1)}\,, \qquad k = 1, \cdots, p+1\,. \tag{3.17}
$$

In the above equation δ is the Čech coboundary operator, $d_{-1}\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}$ is the injection of numbers into $\Omega^{(0)}(M)$ and $\tilde{d}$ the differential of the Deligne complex (it coincides with the de Rham differential d, up to degree $p-1$ and is the **zero** map at degree p). By convention, cohomology (resp. homology) indices are upper (resp. lower) indices, those referring to Čech complex coming first.

Note that $\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}$ is necessarily a cocycle, and, although $\tilde{d}\omega^{(0,p)} \equiv 0$, $d\omega^{(0,p)}$ is the restriction of a globally defined closed form $\omega^{(-1,p+1)}$ with integral periods [25]. This $\omega^{(-1,p+1)}$ will be called the top form of the cocycle $\omega_{\mathcal{D}}^{(p)}$.

We can now proceed and build Deligne-Beilinson cohomology classes as equivalence classes of Deligne-Beilinson cocycles related as follows:

$$
\varpi_{\mathcal{D}}^{(p)} = \omega_{\mathcal{D}}^{(p)} + D\,Q_{\mathcal{D}}\,,
$$

i.e.

$$
\begin{aligned}
\varpi^{(0,p)} &= \omega^{(0,p)} + dq^{(0,p-1)}\,, \\
\varpi^{(k,p-k)} &= \omega^{(k,p-k)} + dq^{(k,p-k-1)} + \delta q^{(k-1,p-k)}\,, \qquad k = 1, \ldots, p\,, \\
\overset{\mathbb{Z}}{\varpi}{}^{(p+1,-1)} &= \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} + \delta\overset{\mathbb{Z}}{q}{}^{(p,-1)}\,,
\end{aligned} \tag{3.18}
$$

where $q^{(k,p-k-1)} \in \check{\mathrm{C}}^k(\mathcal{U},\Omega^{(p-k-1)}(M))$ and $\overset{\mathbb{Z}}{q}{}^{(p,-1)} \in \check{\mathrm{C}}^p(\mathcal{U})$. As an immediate consequence, all cocycles belonging to the same Deligne-Beilinson cohomology class have the same top form.

The integral of a Deligne-Beilinson cocycle $\omega_{\mathcal{D}}^{(p)}$ over a p-cycle $z_{(p)}^{\mathcal{W}}$ is naturally defined as the pairing

$$\begin{aligned}\int_{z_{(p)}^{\mathcal{W}}} \omega_{\mathcal{D}}^{(p)} \;&:=\; \left\langle \omega_{\mathcal{D}}^{(p)}, z_{(p)}^{\mathcal{W}} \right\rangle \;:=\; \sum_{k=0}^{p} \int_{z_{(k,p-k)}} \omega^{(k,p-k)} \\ &:= \sum_{k=0}^{p} \frac{1}{(k+1)!} \sum_{\alpha_0,\cdots,\alpha_k} \int_{z_{(k,p-k),\alpha_0\cdots\alpha_k}} \omega_{\alpha_0\cdots\alpha_k}^{(k,p-k)} \,. \end{aligned} \tag{3.19}$$

In (3.19) the ambiguities on the representatives $z_{(p)}^{\mathcal{W}}$ (resp. $\omega_{\mathcal{D}}^{(p)}$) of $[z_{(p)}^{\mathcal{W}}]$ (resp. $[\omega_{\mathcal{D}}^{(p)}]$) generate terms of the form

$$\int_{h_{(p+1,0)}} d_{-1}\left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} + \delta \overset{\mathbb{Z}}{q}{}^{(p,-1)}\right) \;+\; \int_{z_{(p,0)}} d_{-1}\overset{\mathbb{Z}}{q}{}^{(p,-1)} \,. \tag{3.20}$$

These terms are necessarily integers since the chains and the cochains appearing there are integers. In other words, (3.19) extends to classes as long as we work modulo "integers". This also means that the duality so realized is over $\mathbb{R}/\mathbb{Z}$, not $\mathbb{R}$, i.e. of Pontrjagin type. Actually, this is not totally surprising since a Deligne-Beilinson cohomology class defines a form up to a form with integral periods (cf. appendix A).

Many of the equalities we will encounter only hold true *mod* $\mathbb{Z}$, accordingly we shall use the notation " $\overset{\mathbb{Z}}{=}$ " to mean " $= \;\ldots$ mod $\mathbb{Z}$".

With all this information, we finally set

$$\int_{z_p} \left[\omega_{\mathcal{D}}^{(p)}\right] := \int_{\left[z_{(p)}^{\mathcal{W}}\right]} \left[\omega_{\mathcal{D}}^{(p)}\right] \;\overset{\mathbb{Z}}{=}\; \sum_{k=0}^{p} \int_{z_{(k,p-k)}} \omega^{(k,p-k)} \,, \tag{3.21}$$

for any representative of $[\omega_{\mathcal{D}}^{(p)}]$ and $[z_{(p)}^{\mathcal{W}}]$ to which we shall refer to (3.21) as the *"Defining Formula"*.

Let us note that the linearity of (3.21) with respect to z_p is clear since all descents are linear.

3.2.1 Examples

Let us apply (3.21) to two simple cases. First, consider the situation where the cycle z_p is a boundary: $z_p = bc_{p+1}$. Due to the equivalence of singular and Čech homologies, any Čech p-cycle, $z_{(p,-1)}$, arising from the descent of z_p, is a Čech boundary, i.e.

$$z_{(p,-1)} = \partial c_{(p+1,-1)} \,, \tag{3.22}$$

for some integral Čech chain $c_{(p+1,-1)}$. Then, the corresponding descent has a representative of the form

$$z_{(p)}^{\mathcal{W}} := (z_{(0,p)} = bc_{(0,p+1)}, 0\ldots, 0, \ldots, 0) \,, \tag{3.23}$$

with $\partial c_{(0,p+1)} = c_{p+1}$. Accordingly, the integral of $[\omega_{\mathcal{D}}^{(p)}]$ over this trivial cycle z_p reads

$$\begin{aligned}\int_{z_p} \left[\omega_{\mathcal{D}}^{(p)}\right] &\overset{\mathbb{Z}}{=} \int_{bc_{(0,p+1)}} \omega^{(0,p)} \overset{\mathbb{Z}}{=} \int_{c_{(0,p+1)}} d\omega^{(0,p)} \overset{\mathbb{Z}}{=} \int_{c_{(0,p+1)}} \delta_{-1}\omega^{(-1,p+1)} \\ &\overset{\mathbb{Z}}{=} \int_{\partial c_{(0,p+1)}} \omega^{(-1,p+1)} \overset{\mathbb{Z}}{=} \int_{c_{p+1}} \omega^{(-1,p+1)} . \end{aligned} \tag{3.24}$$

This property is exactly what we were expecting when we considered the integration of a U(1)-connection (cf the introduction to this section).

Second, let us assume that the $(p+1)$-form associated to $[\omega_{\mathcal{D}}^{(p)}]$ is exact. Then, it can be shown that there is a Deligne-Beilinson representative

$$\omega_{\mathcal{D}}^{(p)} := \left(\omega^{(0,p)} = \delta_{-1} q^{(-1,p)}, 0 \ldots, 0, \ldots, 0\right) \tag{3.25}$$

of $[\omega_{\mathcal{D}}^{(p)}]$, where $\omega^{(-1,p+1)} = dq^{(-1,p)}$. The integration formula now reads

$$\begin{aligned}\int_{z_p} \left[\omega_{\mathcal{D}}^{(p)}\right] &\overset{\mathbb{Z}}{=} \int_{z_{(0,p)}} \omega^{(0,p)} \overset{\mathbb{Z}}{=} \int_{z_{(0,p)}} \delta_{-1} q^{(-1,p)} \\ &\overset{\mathbb{Z}}{=} \int_{\partial z_{(0,p)}} q^{(-1,p)} \overset{\mathbb{Z}}{=} \int_{z_p} q^{(-1,p)} , \end{aligned} \tag{3.26}$$

as expected. Indeed, on the one hand, as we write $\omega^{(-1,p+1)} = dq^{(-1,p)}$, we canonically associate to $\omega^{(-1,p+1)}$ a definite form, on the other hand, we have emphasized the fact that a Deligne-Beilinson cohomology class $[\omega_{\mathcal{D}}^{(p)}]$ defines a p-form on M, up to p-forms with integral periods, $q^{(-1,p)}$. It is then natural to find that the integral of $[\omega_{\mathcal{D}}^{(p)}]$ over a cycle coincides -up to integers- with the integral of $q^{(-1,p)}$ over this cycle.

3.3 An equivalent integration over the whole manifold

In the previous approach that led to the *Defining Formula*, we have only dealt with integrals defined over cycles. In view of further generalization we shall first express those as integrals over the whole manifold M. A way to do so is to construct a version of Pontrjagin duality in the Deligne-Beilinson framework. In other words, we construct a (non smooth) canonical Deligne-Beilinson cohomology class $[\eta_{\mathcal{D}}^{(n-p-1)}(z)]$ associated to any singular p-cycle z on M and a cup product $(\cup_{\mathcal{D}})$ [6, 11, 13] such that

$$\int_z \left[\omega_{\mathcal{D}}^{(p)}\right] \overset{\mathbb{Z}}{=} \int_M \left[\omega_{\mathcal{D}}^{(p)}\right] \cup_{\mathcal{D}} \left[\eta_{\mathcal{D}}^{(n-p-1)}(z)\right] , \tag{3.27}$$

for any Deligne-Beilinson cohomology class $[\omega_{\mathcal{D}}^{(p)}]$. We refer the reader to appendix B for a construction of (a representative of) $[\eta_{\mathcal{D}}^{(n-p-1)}(z)]$. Now, let

$$\eta_{\mathcal{D}}^{(n-p-1)}(z) := \left(\eta^{(0,n-p-1)}, \ldots, \eta^{(n-p-1,0)}, \overset{\mathbb{Z}}{\eta}{}^{(n-p,-1)}\right) ,$$

be a representative of $[\eta_{\mathcal{D}}^{(n-p-1)}(z)]$ and

$$\omega_{\mathcal{D}}^{(p)} := \left(\omega^{(0,p)}, \ldots, \omega^{(p,0)}, \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}\right) ,$$

a representative of a Deligne-Beilinson cohomology class $[\omega_{\mathcal{D}}^{(p)}]$. Then a representative of the cup product $[\omega_{\mathcal{D}}^{(p)}] \cup_{\mathcal{D}} [\eta_{\mathcal{D}}^{(n-p-1)}(z)]$ is given by

$$\Big(\omega^{(0,p)} \cup d\eta^{(0,n-p-1)}, \ldots, \omega^{(p,0)} \cup d\eta^{(0,n-p-1)}, \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(0,n-p-1)}, \ldots, \\ \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(n-p-1,0)}, \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \overset{\mathbb{Z}}{\eta}{}^{(n-p,-1)}\Big). \tag{3.28}$$

The cup product $\cup$ within the Čech-de Rahm complex is defined in [27]. In this Deligne-Beilinson cohomology class, $\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \overset{\mathbb{Z}}{\eta}{}^{(n-p,-1)}$ is an integral Čech $(n+1)$-cocycle which is necessarily trivial since the covering of M is simple. Hence

$$\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \overset{\mathbb{Z}}{\eta}{}^{(n-p,-1)} = \delta \overset{\mathbb{Z}}{\chi}{}^{(n,-1)} \tag{3.29}$$

for some integral Čech n-cochain $\overset{\mathbb{Z}}{\chi}{}^{(n,-1)}$. Accordingly, considering M itself as a cycle we can associate to it a Weil decomposition [9]

$$M^{\mathcal{W}} = (m_{(0,n)}, \ldots, m_{(k,n-k)}, \ldots, m_{(n,0)}), \tag{3.30}$$

so that we obtain

$$\begin{aligned}\int_M \left[\omega_{\mathcal{D}}^{(p)}\right] \cup_{\mathcal{D}} \left[\eta_{\mathcal{D}}^{(n-p-1)}(z)\right] &\overset{\mathbb{Z}}{=} \sum_{k=0}^{p} \int_{m_{(k,n-k)}} \omega^{(k,p-k)} \cup d\eta^{(0,n-p-1)} + \\ &\quad + \sum_{k=p+1}^{n} \int_{m_{(k,n-k)}} \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(k-p-1,n-k)} \\ &\overset{\mathbb{Z}}{=} \sum_{k=0}^{p} \int_{m_{(k,n-k)}} \omega^{(k,p-k)} \cup d\eta^{(0,n-p-1)}.\end{aligned} \tag{3.31}$$

It has to be noted that not all representatives of $[M^{\mathcal{W}}]$ and of $[\eta_{\mathcal{D}}^{(n-p-1)}(z)]$ are suitable. Indeed, representatives of $[\eta_{\mathcal{D}}^{(n-p-1)}(z)]$ are de Rham currents and so cannot always be integrated on a singular chain. Strictly speaking, the integration is possible only when the current and the chain are transversal; this is the same problem as encountered in trying to define the product of distributions. Intersection theory of chains in $\mathbb{R}^n$ assures that there exist representatives of $[M^{\mathcal{W}}]$ and $[z_{(p)}^{\mathcal{W}}]$ for which (3.31) is well defined. More precisely the allowed ambiguities on the representatives of the m's and the η's are just those required to set the chains they represent in *"general position"*, so that their intersection can be defined (see for instance [26]). Then we can show that (3.31) gives, up to integers, the same result as (3.21).

We shall refer to formula (3.31) as the "*Long Formula*" which obviously allows to generalize the integration of $[\omega_{\mathcal{D}}^{(p)}]$ over cycles in the sense that we can now define the Deligne-Beilinson product of $[\omega_{\mathcal{D}}^{(p)}]$ with any Deligne-Beilinson cohomology class $[\eta_{\mathcal{D}}^{(n-p-1)}]$ (not necessarily representing a singular cycle) and integrate over M. As an exercise, one can check that the two simple cases presented in subsection (3.2.1) lead to the same results when using the *Long Formula*, instead of the *Defining Formula*.

[9]Which is nothing but a polyhedral decomposition of M, as defined in [22].

3.4 Smoothing

Instead of using singular chains as in the previous construction we use here de Rham chains which are equivalence classes of singular chains — *for which the integrals of any smooth form on M are the same ([22, p. 28])* —. Accordingly we introduce de Rham integration currents

$$T(z)^{n-p+k}_{\ k}$$

associated with $z_{(k,p-k)}$, elements of which can be seen as $(n-p)$-forms with compact supports (and distributional coefficients). In analogy with (3.10) we obtain a sequence of currents

$$T^{(p)}_{\mathcal{W}}(z) = (T(z)^{n-p}_{\ 0}, \ldots, T(z)^{n-p+k}_{\ k}, \ldots, T(z)^{n}_{p}\ ), \tag{3.32}$$

and the descent equations

$$dT(z)^{n-p+k}_{\ k} = \partial T(z)^{n-p+k+1}_{\ k+1}, \tag{3.33}$$

for $k = 1, \ldots, p-1$ and

$$\partial T(z)^{n-p}_0 = T(z)^{n-p}_{-1} := T(z) \qquad \int_M T(z)^n_p \in [z_{(p,-1)}], \tag{3.34}$$

where $T(z)$ is the integration current of z and $[z_{(p,-1)}]$ is the Čech homology class of z in M. In terms of these de Rham currents, the *Defining Formula* reads

$$\int_z \left[\omega^{(p)}_{\mathcal{D}}\right] \stackrel{\mathbb{Z}}{=} \sum_{k=0}^{p} \int_M T(z)^{n-p+k}_k \odot \omega^{(k,p-k)}, \tag{3.35}$$

where we define:

$$T(z)^{n-p+k}_k \odot \omega^{(k,p-k)} = \frac{1}{(k+1)!} \sum_{\alpha_0,\cdots,\alpha_k} T(z)^{n-p+k}_{k,\,\alpha_0\cdots\alpha_k} \wedge \omega^{(k,p-k)}_{\alpha_0\cdots\alpha_k}. \tag{3.36}$$

As a special case, the whole cycle M gives rise to a sequence

$$T_{\mathcal{W}}(M) = \left(T(M)^0_0, \ldots, T(M)^k_k, \ldots, T(M)^n_n\right), \tag{3.37}$$

with

$$dT(M)^k_k = \partial T(M)^{k+1}_{k+1} \tag{3.38}$$

for $k = 1, \ldots, n-1$ and

$$\partial T(M)^0_0 = T(M)^{\ 0}_{-1} := T(M) = 1; \qquad \int_M T(M)^n_n \in [m_{(n,-1)}], \tag{3.39}$$

$[m_{(n,-1)}]$ being the Čech homology class of M. Accordingly, the *Long Formula* now reads

$$\begin{aligned}\int_M \left[\omega^{(p)}_{\mathcal{D}}\right] \cup_{\mathcal{D}} \left[\eta^{(n-p-1)}_{\mathcal{D}}(z)\right] \stackrel{\mathbb{Z}}{=} & \sum_{k=0}^{p} \int_M T(M)^k_k \odot \left(\omega^{(k,p-k)} \cup d\eta^{(0,n-p-1)}\right) + \\ & + \sum_{k=p+1}^{n} \int_M T(M)^k_k \odot \left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(k-p-1,n-k)}\right).\end{aligned} \tag{3.40}$$

The allowed ambiguities of de Rham currents representing $[T_{\mathcal{W}}(M)]$ are bigger than those implied by the Weil descent in the decomposition of M, except at the first and the last steps — cf. (3.39) —. Indeed, in (3.40) an ambiguity may be any de Rham current and not necessarily the integration current of an integral chain as in the case of (3.30), in particular it can be any **smooth** form (but still with compact support). This freedom on the ambiguities allows us to smooth the $T(M)^k_k$ currents occurring in the *Long Formula*, replacing them by differential forms induced by a partition of unity on M, as shown below.

Let us seek for sequences of (smooth) forms that satisfy the same descent equations as $T_{\mathcal{W}}(M)$ and such that when substituted into (3.40) they define the same integrals. Concerning the descent equations, it is well-known (see for instance [25]) that a partition of unity on M, subordinate to the simple covering $\mathcal{U}$ of M, gives rise to a sequence of forms

$$\Theta_{\mathcal{W}}(M) := (\vartheta^0_0, \ldots, \vartheta^k_k, \ldots, \vartheta^n_n\ )\,, \tag{3.41}$$

which satisfy homological descent equations

$$d\vartheta^k_k = \partial\vartheta^{k+1}_{k+1} \tag{3.42}$$

$k = 1, \ldots, n-1$, as well as

$$\partial\vartheta^0_0 = \vartheta^{-1}_{-1} = 1\,. \tag{3.43}$$

Furthermore, since M is supposed to be compact, the forms ϑ^k_k can all be chosen with compact supports in their defining open sets. Due to the smoothness of all the components of $\Theta_{\mathcal{W}}(M)$, the second constraint of (3.39) reads

$$\int_M \vartheta^n_n := t_{(n,-1)} + \partial r_{(n+1,-1)}\,, \tag{3.44}$$

where $t_{(n,-1)}$ is an integral Čech cycle while $r_{(n+1,-1)}$ is a **real** Čech chain. That is to say, ϑ^n_n defines an integral cycle up to a real boundary. Using homological and cohomological descents, one can show that $t_{(n,-1)} \in [m_{(n,-1)}]$. This is mainly due to the fact that the integration of any closed n-form on M can be performed by means of a partition of unity on M.

Let us compare $T_{\mathcal{W}}(M)$ with $\Theta_{\mathcal{W}}(M)$ in order to replace $T_{\mathcal{W}}(M)$ by $\Theta_{\mathcal{W}}(M)$ in (3.40). To begin with,

$$\partial\vartheta^0_0 - \partial T^0_0 = 0 \quad \Rightarrow \quad \vartheta^0_0 - T^0_0 = \partial R^0_1\ + d_{-1}R^{-1}_0\,, \tag{3.45}$$

with $\partial d_{-1}R^{-1}_0 = 0$, hence $\partial R^{-1}_0 = 0$. As M is connected $H_0(M,\mathbb{R}) = 0$, $R^{-1}_0 = \partial R^{-1}_1$. T^0_0 can be replaced by ϑ^0_0 in (3.40) since

$$\int_M d_{-1}R^{-1}_0 \odot \left(\omega^{(0,p-0)} \cup d\eta^{(0,n-p-1)}\right) = \int_M d\left[R^{-1}_1 \odot \left(\omega^{(0,p-0)} \cup d\eta^{(0,n-p-1)}\right)\right] = 0\,. \tag{3.46}$$

Thus R^{-1}_0 can be ignored in (3.45) and the first step of the descent reads

$$\partial(\vartheta^1_1 - T^1_1) = d(\vartheta^0_0 - T^0_0) = d\partial R^0_1 = \partial d R^0_1\,, \tag{3.47}$$

so that

$$\vartheta^1_1 - T^1_1 = dR^0_1 + \partial R^1_2\,. \tag{3.48}$$

Similarly

$$\vartheta_k^k - T_k^k = dR_k^{k-1} + \partial R_{k+1}^k \,, \qquad k = 1, \ldots, n. \tag{3.49}$$

Finally, the constraints (3.34) and (3.44) give

$$\int_M (\vartheta_n^n - T_n^n) = \partial \lambda_{n+1}^{-1} = \partial \int_M R_{n+1}^n \,. \tag{3.50}$$

Now, if we replace T_k^k by ϑ_k^k and its ambiguities, the *Long Formula* reads:

$$\begin{aligned} \int_M (\ldots) \; \stackrel{\mathbb{Z}}{=} \; & \sum_{k=0}^{p} \int_M \vartheta_k^k \odot \left(\omega^{(k,p-k)} \cup d\eta^{(0,n-p-1)} \right) + \\ & + \sum_{k=p+1}^{n} \int_M \vartheta_k^k \odot \left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(k-p-1,n-k)} \right) + \\ & + \int_M \partial R_{n+1}^n \odot \left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(n-p-1,0)} \right) . \end{aligned} \tag{3.51}$$

The last term in this equation gives

$$\begin{aligned} \int_M R_{n+1}^n \odot \delta \, (\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(n-p-1,0)}) &= \int_M R_{n+1}^n \odot \delta \overset{\mathbb{Z}}{\chi}{}^{(n,-1)} \\ &= \int_M \partial R_{n+1}^n \odot \overset{\mathbb{Z}}{\chi}{}^{(n,-1)} = \int_M (\vartheta_n^n - T_n^n) \odot \overset{\mathbb{Z}}{\chi}{}^{(n,-1)} \end{aligned} \tag{3.52}$$

Since all integrals of T_n^n's are integers, we obtain

$$\int_M R_{n+1}^n \odot \delta \left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(n-p-1,0)} \right) \stackrel{\mathbb{Z}}{=} \int_M \vartheta_n^n \odot \overset{\mathbb{Z}}{\chi}{}^{(n,-1)} \,,$$

so that the *(smoothed) Long Formula* reads

$$\begin{aligned} \int_z \left[\omega_{\mathcal{D}}^{(p)} \right] \; \stackrel{\mathbb{Z}}{=} \; & \sum_{k=0}^{p} \int_M \vartheta_k^k \odot \left(\omega^{(k,p-k)} \cup d\eta^{(0,n-p-1)} \right) \\ & + \sum_{k=p+1}^{n} \int_M \vartheta_k^k \odot \left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(k-p-1,n-k)} \right) + \int_M \vartheta_n^n \odot \overset{\mathbb{Z}}{\chi}{}^{(n,-1)} . \end{aligned} \tag{3.53}$$

Let us make some final remarks. First, if the simple covering $\mathcal{U}$ of M is such that all intersections of order larger than $n+1$ are empty — we shall say that $\mathcal{U}$ is *"excellent"* — we deduce that

$$\int_M \vartheta_n^n \odot \overset{\mathbb{Z}}{\chi}{}^{(n,-1)} \in \mathbb{Z} \,, \tag{3.54}$$

which leads to

$$\begin{aligned} \int_z [\omega_{\mathcal{D}}^{(p)}] \; \stackrel{\mathbb{Z}}{=} \; & \sum_{k=0}^{p} \int_M \vartheta_k^k \odot \left(\omega^{(k,p-k)} \cup d\eta^{(0,n-p-1)} \right) + \\ & + \sum_{k=p+1}^{n} \int_M \vartheta_k^k \odot \left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(k-p-1,n-k)} \right) . \end{aligned} \tag{3.55}$$

In other words, with respect to an excellent covering of M, the ϑ_k^k's play the role of the integration currents $T(M)_k^k$ of the Weil descent of M.

Second, the previous construction, i.e. the smoothing, cannot be applied to the *Defining Formula* without care. Indeed, a simple covering of M does not always induce a simple covering on the p-cycle z_p, so that, although a $(p+1)$-cocycle on M reduces to a $(p+1)$-cocycle on z_p, this cocycle is not necessarily trivial. Therefore we cannot establish a smoothed *Defining Formula* in full generality. However, let us assume that z_p admits a tubular neighborhood, $\mathcal{V}_z$, such that $\mathcal{U}_{|\mathcal{V}_z}$ - the restriction to $\mathcal{V}_z$ of the simple covering $\mathcal{U}$ of M - is also simple. Then, as a tubular neighborhood, $\mathcal{V}_z$ has necessarily the same cohomology as z_p, and since $\mathcal{U}_{|\mathcal{V}_z}$ is simple, this cohomology is also the Čech cohomology of $\mathcal{U}_{|\mathcal{V}_z}$. In particular the Čech $(p+1)$-cocycle, $\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}$, on M is also a $(p+1)$-cocycle on $\mathcal{V}_z$ and is necessarily trivial on it, that is: $\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} = \delta \overset{\mathbb{Z}}{\varpi}{}^{(p,-1)}$ for some integral Čech p-cochain $\overset{\mathbb{Z}}{\varpi}{}^{(p,-1)}$, just as in the case of the *Long Formula*. With all this, a natural candidate for a smoothed *Defining Formula* would be

$$\int_{z_p} \left[\omega_{\mathcal{D}}^{(p)}\right] \overset{\mathbb{Z}}{=} \sum_{k=0}^{p-1} \int_{z_p} \vartheta_k^k \odot \omega^{(k,p-k)} + \int_{z_p} \vartheta_p^p \odot \left(\omega^{(p,0)} - \overset{\mathbb{Z}}{\varpi}{}^{(p,-1)}\right),$$

which compares to the smoothed *Long Formula* (3.53).

As a third remark, one can wonder what is the relation between the *Defining Formulas* and the decomposition $A = A_0 + \alpha$ used in section 2 in the case of U(1)-connections. Let us consider two Deligne-Beilinson classes, $[\omega_{\mathcal{D}}]$ and $[\chi_{\mathcal{D}}]$, representing the same Čech cohomology class, $[\check{\xi}]$, as detailed in appendix A.6. We know that $[\omega_{\mathcal{D}}]$ and $[\chi_{\mathcal{D}}]$ differ by a Deligne-Beilinson class, $[(\delta\alpha)_{\mathcal{D}}]$ coming from a global form α on M. This exactly corresponds to the standard decomposition $A = A_0 + \alpha$ for U(1)-connections met in section 2. This can also be seen at the level of the integrals : choose representatives $(\omega^{(0,p)}, \cdots, \omega^{(p,0)}, \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)})$ and $(\chi^{(0,p)}, \cdots, \chi^{(p,0)}, \overset{\mathbb{Z}}{\chi}{}^{(p+1,-1)})$ of $[\omega_{\mathcal{D}}]$ and $[\chi_{\mathcal{D}}]$ respectively, and write the previous decomposition

$$\begin{aligned}\left(\chi^{(0,p)}, \ldots, \chi^{(p,0)}, \overset{\mathbb{Z}}{\chi}{}^{(p+1,-1)}\right) = \left(\omega^{(0,p)}, \ldots, \omega^{(p,0)}, \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}\right) + (0, \ldots, 0, \delta\alpha) + \\ + D(q^{(0,p-1)}, \cdots, q^{(p-1,0)}, \overset{\mathbb{Z}}{q}{}^{(p,-1)}),\end{aligned} \tag{3.56}$$

for some $[q_{\mathcal{D}}]$. By assumption $\overset{\mathbb{Z}}{\chi}{}^{(p+1,-1)}$ and $\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}$ are cohomologous, so

$$\int_{z_p} [\chi_{\mathcal{D}}] \overset{\mathbb{Z}}{=} \int_{z_p} [\omega_{\mathcal{D}}] + \int_{z_p} \alpha\,.$$

This result also means that the standard decomposition $A = A_0 + \alpha$ of U(1)-connections, extends to any generalized p-connection.

A final remark on notations, we could have denoted the integral over z_p of the class $[\omega_{\mathcal{D}}^{(p)}]$ simply as:

$$\begin{aligned}\int_{z_p} \left[\omega_{\mathcal{D}}^{(p)}\right] &\overset{\mathbb{Z}}{=} \left\langle \left[\omega_{\mathcal{D}}^{(p)}\right], \left[z_{(p)}^{\mathcal{W}}\right]\right\rangle \\ &\overset{\mathbb{Z}}{=} \left\langle \omega^{(0,p)} + \cdots \omega^{(p,0)} + \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}, z_{(0,p)} + \cdots + z_{(p,0)}\right\rangle \\ &\overset{\mathbb{Z}}{=} \left\langle [\omega_{\mathcal{D}}^{(p)}] \cup_{\mathcal{D}} [\eta_{\mathcal{D}}^{(n-p-1)}(z)],\ M\right\rangle\end{aligned} \tag{3.57}$$

JHEP08(2005)027

which has the advantage to make easier the proof of independence with respect to the various representatives.

4. Integration of Deligne-Beilinson classes with distributional coefficients

In any quantization procedure, ω will be by nature distributional and integration over a cycle will, in general, be ill defined so that the integration current of the cycle will have to be replaced by some regularized form. This is the situation which has been exhibited in the example of section 2. A canonical way to perform such an operation for $[\omega_{\mathcal{D}}^{(p)}]$ of distributional character is to use formula (3.53), (3.57) with $z_{(p)}^{\mathcal{W}}$ replaced by a smooth Deligne-Beilinson class $[j_{\mathcal{D}}^{(n-p-1)}]$, the integration formula being

$$\begin{aligned}\left\langle [\omega_{\mathcal{D}}^{(p)}] \cup_{\mathcal{D}} [j_{\mathcal{D}}^{(n-p-1)}],\ M \right\rangle \overset{\mathbb{Z}}{=} & \sum_{k=0}^{p} \int_M \vartheta_k^k \odot \left(\omega^{(k,p-k)} \cup dj^{(0,n-p-1)} \right) + \\ & + \sum_{k=p+1}^{n} \int_M \vartheta_k^k \odot \left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup j^{(k-p-1,n-k)} \right) \\ & + \int_M \vartheta_n^n \odot \overset{\mathbb{Z}}{\chi}{}^{(n,-1)} \,. \end{aligned} \tag{4.1}$$

Note that (4.1) is — **mod** $\mathbb{Z}$! — symmetric in $[\omega_{\mathcal{D}}^{(p)}]$ and $[j_{\mathcal{D}}^{(n-p-1)}]$, as can be easily verified. Whereas we have shown that to the current of a cycle z_p is associated a special Deligne-Beilinson class $[\eta_{\mathcal{D}}^{(n-p-1)}(z)]$, the map $z_p \to [\eta_{\mathcal{D}}^{(n-p-1)}(z)]$ being analogous to the cycle map in [6], we do not know of such an assignment in the case of a smoothed version. It is expected that after renormalization some of the characteristics of the regularized class $[j_{\mathcal{D}}^{(n-p-1)}]$ will survive.

5. Conclusions

We have described in some details a class of topological actions which are "topological" in the sense that they are defined modulo "integers", a situation repeatedly met in semi classical treatments of various field theories involving particular geometries (mostly gauge theories, including gravity). They are described by integral formulae which involve refinements of closed differential forms with integral periods named Deligne-Beilinson cohomology classes. The integrals are written as pairings of two such classes in such a way that one of them may have a distributional character as demanded in most field theory contexts.

A. Deligne-Beilinson cohomology

We have not been able to find an elementary discussion of Deligne-Beilinson cohomology in the mathematical literature. The purpose of this appendix is to fill in this gap, concentrating on the computation of Deligne-Beilinson cohomology and on the proof of its independence upon the covering. For more algebraic exposés we refer to [11, 13].

JHEP08(2005)027

A.1 Definitions and notations

As in the main text, M denotes a compact differentiable manifold of dimension n, and $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ a simple covering[10] of M, $M = \cup_{\alpha\in I}\mathcal{U}_\alpha$. A Čech cochain of degree k with values in an abelian group G is a collection of elements $c_{\alpha_0\cdots\alpha_k}$ of G, one for each intersection $\mathcal{U}_{\alpha_0\cdots\alpha_k}$, which is totally antisymmetric in all its indices and vanishes on empty intersections. A Čech cochain of degree -1 is a constant map from M to G.

The Čech differential, δ, maps $(k-1)$-cochains to k-cochains and squares to 0. Acting on (-1)-cochains, δ is the restriction : $(\delta c)_{\alpha_0} = c$ on any non empty $\mathcal{U}_{\alpha_0}$. For $k \geq 1$, if $c_{\alpha_0\cdots\alpha_{k-1}}$ is a $(k-1)$-cochain and $\mathcal{U}_{\alpha_0\cdots\alpha_k} \neq \emptyset$,

$$(\delta c)_{\alpha_0\cdots\alpha_k} = \sum_{i=0}^{k} (-)^i c_{\alpha_0\cdots\widehat{\alpha}_i\cdots\alpha_k} \tag{A.1}$$

were the $\widehat{}$ means omission.

The elements in the kernel of δ are Čech cocycles, those in the image of δ are Čech coboundaries.

In the sequel, we shall have no use of general abelian groups G, but $\mathbb{R}$ (for real Čech cochains), $\mathbb{Z}$ (for integral Čech cochains) and $\mathbb{R}/\mathbb{Z}$ will play preferred roles.

One can also consider Čech cochains where each $c_{\alpha_0\cdots\alpha_k}$ is a differential l-form defined on $\mathcal{U}_{\alpha_0\cdots\alpha_k}$; such cochains are often referred to as Čech-de Rham cochains of bidegree (k,l). In Čech degree -1, we retrieve global differential l-forms defined on M and δ is still defined by restriction. On these "extended" Čech $(k-1)$-cochains, $k \geq 1$, the action of δ is still given by (A.1) except for an overall multiplicative factor $(-)^{l+1}$ on the right hand side : each term makes sense with the proviso that it is restricted to the corresponding $(k+1)$-fold intersection . This leads to the space[11] denoted by $\check{C}^{(k)}(\mathcal{U},\Omega^l(M))$ in the main text. To save space in this appendix, we shall denote it simply by $\Omega^{(k,l)}(\mathbb{R})$, because most of the time M and $\mathcal{U}$ will be fixed.

By convention, a "purely Čech " cochain with constant coefficients (in a subgroup G of $\mathbb{R}$) receives form degree -1, so it belongs to $\Omega^{(k,-1)}(G)$. The de Rham differential d maps $\Omega^{(k,l)}(G)$ into $\Omega^{(k,l+1)}(G)$ for $k \geq 0$.[12] We extend d to (-1)-forms as the injection which maps an element of $G \subset \mathbb{R}$ to the corresponding constant function. This is sometimes denoted by the symbol d_{-1}. This extension still satisfies $d^2 = 0$.

Later in the appendix, we shall need to compare several simple coverings. Suppose that the simple covering $\mathcal{V} = \{\mathcal{V}_\sigma\}_{\sigma\in J}$ of M is a refinement of the simple covering $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha\in I}$: this means that there is the restriction map $r : J \longrightarrow I$ such that $\mathcal{V}_\sigma \subset \mathcal{U}_{r(\sigma)}$ for all indices $\sigma \in J$. A Čech k-cochain, c, for $\mathcal{U}$ can be restricted to $\mathcal{V}$: if the intersection $\mathcal{V}_{\sigma_0\cdots\sigma_k}$

[10] Such an open covering is alternatively called a *good* covering in [27]. This means that any finite intersection of $\mathcal{U}_\alpha$'s, $\mathcal{U}_{\alpha_0\cdots\alpha_q} = \mathcal{U}_{\alpha_0} \cap \cdots \cap \mathcal{U}_{\alpha_q}$, $(\alpha_0,\cdots,\alpha_q) \in I^{q+1}$, is either empty or diffeomorphic to $\mathbb{R}^n$.

[11] A more appropriate language for this setting involves sheaves, but we shall not use the corresponding terminology.

[12] The sign factor $(-1)^{l+1}$ insures that $d\,\delta + \delta\,d = 0$.

is nonempty, then so is $\mathcal{U}_{r(\sigma_0)\cdots r(\sigma_k)}$, and

$$r(c)_{\sigma_0\cdots\sigma_k} \equiv \left(c_{r(\sigma_0)\cdots r(\sigma_k)}\right)_{|\mathcal{V}_{\sigma_0\cdots\sigma_k}} .$$

The Čech and de Rham differentials commute with restriction, i.e. $\delta \circ r = r \circ \delta$ (it being understood that the Čech differential on the left-hand side is for the covering $\mathcal{V}$ and on the right-hand side for the covering $\mathcal{U}$) and $d \circ r = r \circ d$.

A.2 Deligne-Beilinson cochains

Take an integer $0 \le p \le n+1$ (n the dimension of the manifold) and consider the double complex

$$\begin{tikzcd}
\Omega^{(0,-1)}(\mathbb{Z}) \arrow[r, "d_{-1}"] \arrow[d, "\delta"] & \Omega^{(0,0)}(\mathbb{R}) \arrow[r, "d"] \arrow[d, "\delta"] & \cdots \arrow[r, "d"] & \Omega^{(0,p-1)}(\mathbb{R}) \arrow[r, "0"] \arrow[d, "\delta"] & 0 \\
\Omega^{(1,-1)}(\mathbb{Z}) \arrow[r, "d_{-1}"] \arrow[d, "\delta"] & \Omega^{(1,0)}(\mathbb{R}) \arrow[r, "d"] \arrow[d, "\delta"] & \cdots \arrow[r, "d"] & \Omega^{(1,p-1)}(\mathbb{R}) \arrow[r, "0"] \arrow[d, "\delta"] & 0 \\
\Omega^{(2,-1)}(\mathbb{Z}) \arrow[r, "d_{-1}"] \arrow[d, "\delta"] & \Omega^{(2,0)}(\mathbb{R}) \arrow[r, "d"] \arrow[d, "\delta"] & \cdots \arrow[r, "d"] & \Omega^{(2,p-1)}(\mathbb{R}) \arrow[r, "0"] \arrow[d, "\delta"] & 0 \\
\vdots & \vdots & & \vdots &
\end{tikzcd}$$

The columns of this diagram form standard Čech complexes. The rows are Deligne complexes of index p, that is de Rham complexes extended to the left by d_{-1} (the injection of integral constants into real functions) and truncated on the right at $(p-1)$-forms by the 0 map. We denote by $\tilde{d}$ this modified differential, to avoid confusion with the de Rham differential, d.

We build a new "diagonal complex" from this double complex. The space C_p^q of Deligne-Beilinson cochains of degree $q \ge 0$ (with fixed index p) is defined by

$$C_p^q = \begin{cases} \Omega^{(q,-1)}(\mathbb{Z}) + \sum_{k=1}^{q} \Omega^{(q-k,k-1)}(\mathbb{R}) & \text{for } 0 \le q < p \\ \Omega^{(p,-1)}(\mathbb{Z}) + \sum_{k=1}^{p} \Omega^{(p-k,k-1)}(\mathbb{R}) & \text{for } \ q = p \\ \Omega^{(q,-1)}(\mathbb{Z}) + \sum_{k=1}^{p} \Omega^{(q-k,k-1)}(\mathbb{R}) & \text{for } q > p \end{cases} .$$

Elements of these spaces are respectively represented by the following sequences :

$$c = \left(c^{(0,q-1)}, \cdots, \overset{\mathbb{Z}}{c}{}^{(q,-1)}\right), \qquad c = \left(c^{(0,p-1)}, \cdots, \overset{\mathbb{Z}}{c}{}^{(p,-1)}\right),$$
$$c = \left(c^{(q-p,p-1)}, \cdots, \overset{\mathbb{Z}}{c}{}^{(q,-1)}\right),$$

with the last element $\mathbb{Z}$-valued.[13]

[13] Our complex contains $\Omega^{(q,-1)}(\mathbb{Z})$ while in the literature one usually finds $\Omega^{(q,-1)}(\mathbb{Z}(p))$, where $\mathbb{Z}(p) = (2i\pi)\mathbb{Z}$. This difference is irrelevant for our purpose.

We set $\mathcal{C}_p = C_p^0 \oplus C_p^1 \oplus \cdots$.

The operator $D = \tilde{d} + \delta$ maps C_p^q to C_p^{q+1} and, due to the sign convention in the definition of δ on l-forms, $D^2 = 0$. The complex $(\mathcal{C}_p, D)$ is called the Deligne-Beilinson complex,[14] and the elements of $\mathcal{C}_p$ Deligne-Beilinson cochains. We write $Z_p^q = \left\{ \mathrm{Ker}\, D : C_p^q \to C_p^{q+1} \right\}$ (resp. $B_p^q = \left\{ \mathrm{Im}\, D : C_p^{q-1} \to C_p^q \right\}$) for the space of Deligne-Beilinson cocycles (resp. coboundaries).

We are interested in the cohomology of $(\mathcal{C}_p, D)$. A priori, it depends on the covering, but we shall see later that the cohomologies for simple coverings are canonically isomorphic.

The projection $\pi : C_p^q \to \Omega^{(q,-1)}(\mathbb{Z})$ gives a chain map

$$\begin{tikzcd}
\cdots \arrow[r, "D"] & C_p^q \arrow[r, "D"] \arrow[d, "\pi"] & C_p^{q+1} \arrow[r, "D"] \arrow[d, "\pi"] & \cdots \\
\cdots \arrow[r, "\delta"] & \Omega^{(q,-1)}(\mathbb{Z}) \arrow[r, "\delta"] & \Omega^{(q+1,-1)}(\mathbb{Z}) \arrow[r, "\delta"] & \cdots
\end{tikzcd}$$

so that in all cases, there is a canonical map $H^q(\mathcal{C}_p, D) \to H^q_{\text{Čech}}(M, \mathbb{Z})$. The computation of $H^q(\mathcal{C}_p, D)$ goes along different lines whether $q \leq p-1$ or $q > p-1$.

A.3 Computation of $H^q(\mathcal{C}_p, D)$, $q < p$

In this case, we use the Poincaré lemma for differential forms (ensuring that for forms of nonnegative degree, the de Rham cohomology is locally trivial) to show that

$$H^q(\mathcal{C}_p, D) \simeq H^{q-1}_{\text{Čech}}(M, \mathbb{R}/\mathbb{Z}) \qquad (I)$$

(the isomorphism is canonical). In particular, the canonical map

$$H^q(\mathcal{C}_p, D) \to H^q_{\text{Čech}}(M, \mathbb{Z})$$

maps $H^q(\mathcal{C}_p, D)$ onto the subgroup $H^q_{\text{Čech}}(M, \mathbb{Z})_{\text{torsion}}$ of torsion classes.

Proof. Suppose $c = (c^{(0,q-1)}, c^{(1,q-2)}, \cdots, c^{(q-1,0)}, \overset{\mathbb{Z}}{c}{}^{(q,-1)})$ is a Deligne-Beilinson cocycle. This implies that $\tilde{d}c^{(0,q-1)} = 0$, and since $q \leq p-1$, the operator $\tilde{d}$ in this equation is the standard de Rham differential. So, by the Poincaré lemma, there is an element $\rho^{(0,q-2)} \in \Omega^{(0,q-2)}(\mathbb{R})$ such that $c^{(0,q-1)} + \tilde{d}\rho^{(0,q-2)} = 0$. Accordingly the cocycle c is cohomologous to the cocycle

$$c + D\rho^{(0,q-2)} = \left(0, \underline{c}^{(1,q-2)}, \cdots, c^{(q-1,0)}, \overset{\mathbb{Z}}{c}{}^{(q,-1)}\right),$$

where $\underline{c}^{(1,q-2)} \equiv c^{(1,q-2)} + \delta\rho^{(0,q-2)}$.

The cocycle condition for $c + D\rho^{(0,q-2)}$ yields $d\underline{c}^{(1,q-2)} = 0$ were d is the standard exterior derivative. The procedure can be iterated to show that the cohomology class of c contains a representative of the form

$$\left(0, \cdots, 0, \underline{c}^{(q-1,0)}, \overset{\mathbb{Z}}{c}{}^{(q,-1)}\right)$$

[14] A better notation would be $(\mathcal{C}_p(M), \mathcal{U}, D)$.

JHEP08(2005)027

with the standard descent equations fulfilled :

$$d\underline{c}^{(q-1,0)} = 0\,, \qquad \delta\underline{c}^{(q-1,0)} = d_{-1}\overset{\mathbb{Z}}{c}{}^{(q,-1)}\,, \qquad \delta\overset{\mathbb{Z}}{c}{}^{(q,-1)} = 0\,.$$

The first equation just tells that $\underline{c}^{(q-1,0)} = d_{-1}\rho^{(q-1,-1)}$, where the components $\rho^{(q-1,-1)}$ are real constants. This, combined with the second equation, implies that the integral Čech cocycle $\overset{\mathbb{Z}}{c}{}^{(q,-1)}$ is exact as a real cocycle, so that it represents a torsion class.

Reduction modulo 1 turns $\rho^{(q-1,-1)}$ into an $\mathbb{R}/\mathbb{Z}$ Čech cocycle and the ambiguity on $\underline{c}^{(q-1,0)}(\text{mod } 1)$ is a Čech coboundary. So we have proved the announced result, (I), which is also the content of the following exact sequence [13]

$$0 \longrightarrow H^{q-1}(M,\mathbb{Z}(p)) \longrightarrow H^{q-1}(M,\mathbb{R}) \longrightarrow H^q(\mathcal{C}_p, D) \longrightarrow H^q(M,\mathbb{Z}(p))_{\text{torsion}} \longrightarrow 0.$$

A.4 The Čech homotopy operator

Here we introduce the Čech homotopy operator that we shall need to compute $H^q(\mathcal{C}_p, D)$ in the special cases $q \geq p$. This homotopy[15] operator, which depends on a partition of unity defined on M, is instrumental to establish the generalized Mayer-Vietoris exact sequence, the Čech-de Rham isomorphism and the *Collating Formula* [27], a construction we illustrate below.

A.4.1 The K operator on the enlarged double complex

Consider the following double complex :

$$\begin{tikzcd}
 & \Omega^{(-1,0)}(\mathbb{R}) \arrow[r, "d"] \arrow[d, "\delta"] & \cdots \arrow[r, "d"] & \Omega^{(-1,p-1)}(\mathbb{R}) \arrow[r, "0"] \arrow[d, "\delta"] & 0 \\
\Omega^{(0,-1)}(\mathbb{Z}) \arrow[r, "d_{-1}"] \arrow[d, "\delta"] & \Omega^{(0,0)}(\mathbb{R}) \arrow[r, "d"] \arrow[d, "\delta"] & \cdots \arrow[r, "d"] & \Omega^{(0,p-1)}(\mathbb{R}) \arrow[r, "0"] \arrow[d, "\delta"] & 0 \\
\Omega^{(1,-1)}(\mathbb{Z}) \arrow[r, "d_{-1}"] \arrow[d, "\delta"] & \Omega^{(1,0)}(\mathbb{R}) \arrow[r, "d"] \arrow[d, "\delta"] & \cdots \arrow[r, "d"] & \Omega^{(1,p-1)}(\mathbb{R}) \arrow[r, "0"] \arrow[d, "\delta"] & 0 \\
\Omega^{(2,-1)}(\mathbb{Z}) \arrow[r, "d_{-1}"] \arrow[d, "\delta"] & \Omega^{(2,0)}(\mathbb{R}) \arrow[r, "d"] \arrow[d, "\delta"] & \cdots \arrow[r, "d"] & \Omega^{(2,p-1)}(\mathbb{R}) \arrow[r, "0"] \arrow[d, "\delta"] & 0 \\
\vdots & \vdots & & \vdots &
\end{tikzcd}$$

where the de Rham complex of global differential forms truncated at degree $(p-1)$ has been added at the top. We extend the definition of D to this enlarged complex.

Let us choose a partition of unity ϑ_α subordinate to the simple covering $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ of M: each ϑ_α is a (smooth) non-negative function on M with compact support in $\mathcal{U}_\alpha$, and $\sum_\alpha \vartheta_\alpha$ is the constant function 1 on M. On the enlarged complex, define an operator K (depending on the chosen partition of unity) as follows.

[15]Ensuring that the Čech cohomology for forms of nonnegative degree is trivial.

JHEP08(2005)027

Take $c = \{c_{\alpha_0\cdots\alpha_k}\} \in \Omega^{(k,l)}(\mathbb{R}),\ k, l \geq 0$. Due to the support properties of the ϑ_α's, $c_{\alpha_0\cdots\alpha_k} \cdot \vartheta_{\alpha_k}$ (extended by 0 outside $\mathcal{U}_{\alpha_k}$) is a smooth differential form in each nonempty $\mathcal{U}_{\alpha_0\cdots\alpha_{k-1}}$. Let $Kc \equiv \{(-)^{l+1} \sum_{\alpha_k} c_{\alpha_0\cdots\alpha_k} \cdot \vartheta_{\alpha_k}\} \in \Omega^{(k-1,l)}(\mathbb{R})$.

For $c \in \Omega^{(-1,l)}(\mathbb{R})$, $l \geq 0$, set $Kc \equiv 0$, and for $c \in \Omega^{(k,-1)}(\mathbb{Z})$, $k \geq 0$, set $Kc \equiv Kd_{-1}c \in \Omega^{(k-1,0)}(\mathbb{R})$.

Though we shall not try to compute its homology, note that $K^2 = 0$ so K is a boundary operator (or equivalently a co-differential).

A.4.2 The homotopy property and the fundamental identity

Algebraic manipulations show that $K\delta + \delta K$ is the identity operator on $\Omega^{(k,l)}(\mathbb{R})$, $k \geq -1, l \geq 0$ and d_{-1} on $\Omega^{(k,-1)}(\mathbb{Z})$, $k \geq 0$. In particular, in the enlarged double complex, the vertical Čech complexes in nonnegative de Rham degree have vanishing Čech cohomology, since K is a homotopy operator.

Acting on the enlarged double complex, $K\tilde{d}$ lowers the Čech degree by one unit, so $K\tilde{d}$ is locally nilpotent and $1 + K\tilde{d}$ is invertible : locally the geometric series for $(1 + K\tilde{d})^{-1}$ stops after a finite number of terms. Moreover, as a consequence of

$$\left(1 + K\tilde{d}\right)\left(\tilde{d} + \delta\right) - \delta(1 + K\tilde{d}) = \tilde{d} + K\tilde{d}\delta - \delta K\tilde{d} = (1 - K\delta - \delta K)\,\tilde{d} = 0\,,$$

(the first equality uses $\tilde{d}^2 = 0$, the second $\tilde{d}\delta = -\delta\tilde{d}$ and the last one that the image of $\tilde{d}$ lives in de Rham degree ≥ 0 where $K\delta + \delta K = 1$) one derives that on the enlarged double complex, D and δ are conjugate, that is

$$(1 + K\tilde{d})D = \delta(1 + K\tilde{d})\,. \qquad (\heartsuit)$$

This fundamental identity ($\heartsuit$) is at the heart of the computation of the Deligne-Beilinson cohomology when $q \geq p$ as shown later in A.5 and A.6. It can also be useful in other contexts as illustrated below.

A.4.3 Relation with the Čech-de Rham isomorphism

Suppose that in the first column of the enlarged complex we replace the coefficient group $\mathbb{Z}$ by $\mathbb{R}$, and that we take $p = n + 1$, n the dimension of the manifold, so that the lines are usual de Rham complexes, hence $\tilde{d} = d$ in this enlarged context and the ($\heartsuit$) identity can be written $(1 + Kd)D = \delta(1 + Kd)$. This double complex is a Čech-de Rham complex with differential $D = d + \delta$ and of course $q < p = n + 1$. In the sequel this is the complex we have in mind when we refer to Čech-de Rham cochains, cocycles or coboundaries.

On the one hand if $c^{(q,-1)} \in \Omega^{(q,-1)}(\mathbb{R})$ is a Čech cocycle, it is a D-cocycle, hence its top component $(-Kd)^{q+1}c^{(q,-1)}$ is a global closed q-form, i.e. a de Rham q-cocycle.

On the other hand if $c^{(q,-1)}$ is a Čech coboundary, $c^{(q,-1)} = \delta\gamma^{(q-1,-1)}$ for some $\gamma^{(q-1,-1)} \in \Omega^{(q-1,-1)}(\mathbb{R})$, then using ($\heartsuit$) $(1+Kd)^{-1}c^{(q,-1)} = D(1+Kd)^{-1}\gamma^{(q-1,-1)}$ is a D-coboundary. Identifying top form components, $(-Kd)^{q+1}c^{(q,-1)}$ is a de Rham coboundary $d(-Kd)^q\,\gamma^{(q-1,-1)}$.

Finally, if $c = (c^{(-1,q)}, \cdots, c^{(q-1,0)}, c^{(q,-1)})$ is a D-cocycle, $c^{(q,-1)}$ is a Čech cocycle, $c^{(-1,q)}$ is a closed global de Rham q-form, and c is D-cohomologous to $(1 + Kd)^{-1}c^{(q,-1)}$. Indeed, start from $D(1+Kd)^{-1}K(c-c^{(q,-1)}) = (1+Kd)^{-1}\delta K(c-c^{(q,-1)})$, a consequence of the ($\heartsuit$) identity. As $c-c^{(q,-1)}$ has no component in de Rham degree -1, $\delta K(c-c^{(q,-1)}) = (1-K\delta)(c-c^{(q,-1)})$ by the homotopy property. By $Dc = 0 = \delta c^{(q,-1)}$, we obtain finally that $\delta K(c-c^{(q,-1)}) = (1+Kd)c - c^{(q,-1)}$. Multiplication by $(1+Kd)^{-1}$ leads to

$$c = (1+Kd)^{-1}c^{(q,-1)} + D(1+Kd)^{-1}K(c-c^{(q,-1)}), \quad (*)$$

proving that c is D-cohomologous to $(1+Kd)^{-1}c^{(q,-1)}$. This implies that $c^{(-1,q)}$ is d-cohomologous to $(-Kd)^{q+1}c^{(q,-1)}$, explicitly,

$$c^{(-1,q)} = (-Kd)^{q+1}c^{(q,-1)} + d\left(K\sum_{r=0}^{q-1}(-dK)^r c^{(r,q-1-r)}\right),$$

which is the famous *Collating Formula*; see e.g. [27], where it is used to prove that $c^{(q,-1)} \to (-Kd)^{q+1}c^{(q,-1)}$ which maps (real) Čech cocycles to de Rham cocycles and (real) Čech coboundaries to de Rham coboundaries induces an isomorphism in cohomology. With notations closer to the ones used in the main text, the *Collating Formula* can be rewritten[16]

$$c^{(-1,q)} = d\left(\vartheta_0^0 \cdot c^{(0,q-1)} + \vartheta_1^1 \cdot c^{(1,q-2)} + \cdots + \vartheta_{q-1}^{q-1} \cdot c^{(q-1,0)}\right) + \vartheta_q^q \cdot c^{(q,-1)}.$$

The *Collating Formula* is related to the Weil theorem which can be rewritten neatly using the Deligne-Beilinson machinery.

First, observe that $C^p_{p+1} = C^p_p$ but $Z^p_{p+1} \subset Z^p_p$. Indeed on $\Omega^{(0,p-1)} \subset C^p_{p+1}$ the operator $\tilde{d}$ is the genuine de Rham differential, while on $\Omega^{(0,p-1)} \subset C^p_p$ it is the 0 map, so the condition to be D-closed is more stringent in the first case. If $c = (c^{(0,p-1)}, c^{(1,p-2)}, \cdots, c^{(0,p-1)}, \overset{\mathbb{Z}}{c}{}^{(p,-1)})$ belongs to Z^p_p, the standard de Rham differential applied to $c^{(0,p-1)}$ leads to a global closed p-form. Indeed, $\delta dc^{(0,p-1)} = d\delta c^{(0,p-1)} = \pm d^2\, c^{(1,p-2)} = 0$, so $dc^{(0,p-1)}$ is the restriction of a global p-form, which is obviously closed. So there is a canonical map $\left\{\mathrm{Ker}\, D : C^p_p \to C^{p+1}_p\right\} \xrightarrow{m} \left\{\mathrm{Ker}\, d : \Omega^{(-1,p)} \to \Omega^{(-1,p+1)}\right\}$. The image of this map is not totally obvious, but this is precisely the content of Weil's theorem [25]: the sequence of abelian groups

$$0 \longrightarrow Z^p_{p+1} \xrightarrow{i} Z^p_p \xrightarrow{m} \left\{\begin{array}{c}\text{Closed global } p\text{-forms}\\ \text{with integral periods}\end{array}\right\} \longrightarrow 0$$

is exact.

A.4.4 Refinements

If the simple covering $\mathcal{V} = \{\mathcal{V}_\sigma\}_{\sigma\in J}$ of M is a refinement of the simple covering $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha\in I}$ and $\{\varphi_\sigma\}$ is a partition of unity for $\mathcal{V}$, we define a (compatible) partition of unity for $\mathcal{U}$ $\{\vartheta_\alpha\} = \{\sum_{\substack{\sigma\in J\\ r(\sigma)=\alpha}} \varphi_\sigma\}$. For compatible partitions of unity, the homotopy operator commutes with restriction, i.e. $K \circ r = r \circ K$ (it being understood that the homotopy operator on the left-hand side is for the covering $\mathcal{V}$ and on the right-hand side for the covering $\mathcal{U}$). To

[16] Cf. (3.41)–(3.43) in the main text for properties of the θ_k^k's.

summarize, restriction commutes with δ, $\tilde{d}$, D and K :

$$\delta \circ r = r \circ \delta\,, \quad \tilde{d} \circ r = r \circ \tilde{d}\,, \quad K \circ r = r \circ K\,, \quad D \circ r = r \circ D\,. \tag{A.2}$$

We could say this more pedantically by drawing the Deligne-Beilinson complexes (or their enlarged versions) for $\mathcal{U}$ and $\mathcal{V}$ on top of each other (in three dimensions) and stating that restriction is a (co-)chain map for all differentials or co-differentials defined up to now.

A.5 Computation of $H^q(\mathcal{C}_p, D)$, $q > p$

We show that for $q > p$,

$$H^q(\mathcal{C}_p, D) \simeq H^q_{\text{Čech}}(M, \mathbb{Z}) \qquad (II)$$

(the isomorphism is canonical).

Proof. Start from the simple observation that for $q > p$ one has the inclusion $K\tilde{d}(C^q_p) \subset C^q_p$, so that one can freely use $(1 + K\tilde{d})D = \delta(1 + K\tilde{d})$ to compute $H^q(\mathcal{C}_p, D)$.

The middle cohomology in the complex $0 \longrightarrow C^{q-1}_p \xrightarrow{\delta} C^q_p \xrightarrow{\delta} C^{q+1}_p \longrightarrow 0$ is concentrated in de Rham degree -1 because δ does not change the de Rham degree and has no cohomology in nonnegative de Rham degree due to the existence of the homotopy operator. So this cohomology is simply $H^q_{\text{Čech}}(M, \mathbb{Z})$. If $\overset{\mathbb{Z}}{c}{}^{(q,-1)} \in \Omega^{(q,-1)}(\mathbb{Z}) \subset C^q_p$ is a Čech cocycle, $(1 + K\tilde{d})^{-1}\ \overset{\mathbb{Z}}{c}{}^{(q,-1)}$ is a D-cocycle. Conversely if the cochain $c = (c^{(q-p,p-1)}, \cdots, \overset{\mathbb{Z}}{c}{}^{(q,-1)}) \in C^q_p$ is a D-cocycle, $\overset{\mathbb{Z}}{c}{}^{(q,-1)}$ is a Čech cocycle and the relation $(*)$ is satisfied i.e.

$$c = (1 + K\tilde{d})^{-1}\ \overset{\mathbb{Z}}{c}{}^{(q,-1)} + D(1 + K\tilde{d})^{-1}K(c - \overset{\mathbb{Z}}{c}{}^{(q,-1)})\,.$$

Hence the projection map $\pi : C^q_p \to \Omega^{(q,-1)}(\mathbb{Z})$ descends to an isomorphism in cohomology which proves the announced result (II).

A.6 The case $q = p$

A full description of $H^p(\mathcal{C}_p, D)$ is complicated in general, but it fits in all cases into an exact sequence of abelian groups[17]

$$0 \longrightarrow \left\{ \begin{array}{c} \text{Closed global } (p-1)\text{-forms} \\ \text{with integral periods} \end{array} \right\} \longrightarrow \Omega^{p-1}(M, \mathbb{R}) \longrightarrow H^p(\mathcal{C}_p, D) \longrightarrow H^p_{\text{Čech}}(M, \mathbb{Z}) \longrightarrow 0 \qquad (III)$$

Proof. We shall treat separately the cases $p = q = 0$ and $p = q \neq 0$, starting with the latter.

Let $c = (c^{(0,p-1)}, \cdots, c^{(p-1,0)}, \overset{\mathbb{Z}}{c}{}^{(p,-1)}) \in C^p_p$ be a D-cocycle, then $\overset{\mathbb{Z}}{c}{}^{(p,-1)}$ is a Čech cocycle and $(*)$ tells us that

$$c - (1 + K\tilde{d})^{-1}\ \overset{\mathbb{Z}}{c}{}^{(p,-1)} = D(1 + K\tilde{d})^{-1}K\left(c - \overset{\mathbb{Z}}{c}{}^{(p,-1)}\right).$$

[17]For instance, when $p = 2$, we recover the classification of line bundles with connection modulo gauge equivalence, as expected. This case is treated in detail in appendix C.

However, we now have $K\tilde{d}(C_p^q) \subset C_p^q + \Omega^{(-1,q)}(\mathbb{R})$, in contrast with the previous case for which we had the inclusion $K\tilde{d}(C_p^q) \subset C_p^q$. Accordingly, as an element of $C_p^{p-1} + \Omega^{(-1,p-1)}(\mathbb{R})$, $(1+K\tilde{d})^{-1}K(c - \overset{\mathbb{Z}}{c}{}^{(p,-1)})$ has a component, say $\gamma^{(-1,p-1)}$, in $\Omega^{(-1,p-1)}(\mathbb{R})$, so we cannot conclude that c and $(1+K\tilde{d})^{-1}\overset{\mathbb{Z}}{c}{}^{(p,-1)}$ are D-cohomologous. Nevertheless $\tilde{d}\gamma^{(-1,p-1)} = 0$ (not d !), hence c is D-cohomologous to $(1+K\tilde{d})^{-1}\overset{\mathbb{Z}}{c}{}^{(p,-1)} + \delta\gamma^{(-1,p-1)}$.

Conversely, the cochain $(1+K\tilde{d})^{-1}\overset{\mathbb{Z}}{c}{}^{(p,-1)} + \delta\gamma^{(-1,p-1)}$ is a Deligne-Beilinson cocycle whenever $\gamma^{(-1,p-1)}$ is a global de Rham $(p-1)$-form and $\overset{\mathbb{Z}}{c}{}^{(p,-1)} \in \Omega^{(p,-1)}(\mathbb{Z})$ is a Čech cocycle.

So we have exhibited a family of "reduced" representatives

$$(1+K\tilde{d})^{-1}\overset{\mathbb{Z}}{c}{}^{(p,-1)} + \delta\gamma^{(-1,p-1)}\,, \qquad (**)$$

of Deligne-Beilinson cohomology classes.

Decomposition $(**)$ leads us to consider the following maps

$$\pi : c = \left(c^{(0,p-1)}, \cdots, c^{(p-1,0)}, \overset{\mathbb{Z}}{c}{}^{(p,-1)}\right) \in C_p^p \;\mapsto \overset{\mathbb{Z}}{c}{}^{(p,-1)} \in \Omega^{(p,-1)}(M,\mathbb{Z}),$$

(already met in subsection A.2), and

$$\phi : \gamma^{(-1,p-1)} \in \Omega^{(-1,p-1)}(\mathbb{R}) \;\mapsto\; (\delta\gamma^{(-1,p-1)}, 0, \ldots, 0) \in C_p^p\,.$$

We provide $\Omega^{(-1,p-1)}$ with the trivial differential $= 0$, so that π and ϕ are maps between complexes. It is quite easy to check that these two maps are chain maps, i.e. $\phi \cdot 0 = D \cdot \phi$ and $\pi \cdot D = \delta \cdot \pi$, hence, passing to cohomology,

$$\Omega^{(-1,p-1)}(\mathbb{R}) \xrightarrow{\hat{\phi}} H^p(\mathcal{C}_p, D) \xrightarrow{\hat{\pi}} H^p_{\text{Čech}}(M,\mathbb{Z})\,.$$

Let us show that $\hat{\pi}$ is surjective. First, by definition and with obvious notations,

$$\hat{\pi}([c]) := \left[\overset{\mathbb{Z}}{c}{}^{(p,-1)}\right].$$

For any class $\xi \in H^p_{\text{Čech}}(M,\mathbb{Z})$, let us pick a representative $\overset{\mathbb{Z}}{c}{}^{(p,-1)}$ of ξ. From $(**)$, we deduce that

$$c = (1+K\tilde{d})^{-1}\overset{\mathbb{Z}}{c}{}^{(p,-1)}$$

is a Deligne-Beilinson cocycle which trivially fulfills $\pi(c) = \overset{\mathbb{Z}}{c}{}^{(p,-1)}$, so that

$$\hat{\pi}([c]) = [\overset{\mathbb{Z}}{c}{}^{(p,-1)}] = \xi\,.$$

This means that any integral Čech cohomology class is the image under $\hat{\pi}$ of a Deligne-Beilinson cohomology class, thus establishing the surjectivity of $\hat{\pi}$.

According to this, we can extend further the previous exact sequence to the right

$$\Omega^{(-1,p-1)}(\mathbb{R}) \xrightarrow{\hat{\phi}} H^p(\mathcal{C}_p, D) \xrightarrow{\hat{\pi}} H^p_{\text{Čech}}(M,\mathbb{Z}) \longrightarrow 0.$$

Now, let us show that this sequence is actually exact on the left, that is to say $\mathrm{Ker}(\hat{\pi}) = \mathrm{Im}(\hat{\phi})$.

JHEP08(2005)027

If $[c] \in \mathrm{Ker}(\hat{\pi})$ then $\hat{\pi}([c]) = [\overset{\mathbb{Z}}{c}{}^{(p,-1)}] = 0$, meaning that any representative of $[\overset{\mathbb{Z}}{c}{}^{(p,-1)}]$ is a Čech coboundary, $\delta\overset{\mathbb{Z}}{\lambda}{}^{(p-1,-1)}$. Thus, if $(1+K\tilde{d})^{-1}\,\overset{\mathbb{Z}}{c}{}^{(p,-1)} + \delta\gamma^{(-1,p-1)}$ is a "reduced" representative of $[c] \in \mathrm{Ker}(\hat{\pi})$, we have

$$c = (1+K\tilde{d})^{-1}\delta\overset{\mathbb{Z}}{\lambda}{}^{(p-1,-1)} + \delta\gamma^{(-1,p-1)} = Dq + \delta\rho^{(-1,p-1)}\,,$$

with $\rho^{(-1,p-1)} = \gamma^{(-1,p-1)} + (-K\tilde{d})^p\overset{\mathbb{Z}}{\lambda}{}^{(p-1,-1)}$. In other words, $\mathrm{Ker}(\hat{\pi})$ is made of Deligne-Beilinson classes $[c]$ that admit a representative of the form $\delta\rho^{(-1,p-1)}$ for some global form $\rho^{(-1,p-1)} \in \Omega^{(-1,p-1)}(\mathbb{R})$. Conversely, for any global form $\rho^{(-1,p-1)} \in \Omega^{(-1,p-1)}(\mathbb{R})$ the Deligne-Beilinson class $[\delta\rho^{(-1,p-1)}]$ trivially belongs to $\mathrm{Ker}(\hat{\pi})$. This implies $\mathrm{Ker}(\hat{\pi}) = \mathrm{Im}(\hat{\phi})$.

So, the sequence

$$\Omega^{(-1,p-1)}(\mathbb{R}) \xrightarrow{\hat{\phi}} H^p(\mathcal{C}_p, D) \xrightarrow{\hat{\pi}} H^p_{\text{Čech}}(M,\mathbb{Z}) \longrightarrow 0\,,$$

is exact, and to extend it to the left, we have to compute $\mathrm{Ker}(\hat{\phi})$.

If $\gamma^{(-1,p-1)} \in \mathrm{Ker}(\hat{\phi})$, then $\hat{\phi}(\gamma^{(-1,p-1)}) = [\delta\gamma^{(-1,p-1)}] = 0$, which means that any representative of $[\delta\gamma^{(-1,p-1)}]$ is a Deligne-Beilinson coboundary. In particular

$$\delta\gamma^{(-1,p-1)} = \left(\delta\gamma^{(-1,p-1)}, 0, \ldots, 0\right) = D\tau\,,$$

for some $\tau = (\tau^{(0,p-2)}, \cdots, \tau^{(p-2,0)}, \overset{\mathbb{Z}}{\tau}{}^{(p-1,-1)}) \in C_p^{p-1}$. This gives rise to the following Čech-de Rham cochain

$$\left(-\gamma^{(-1,p-1)}, \tau^{(0,p-2)}, \cdots, \tau^{(p-2,0)}, \overset{\mathbb{Z}}{\tau}{}^{(p-1,-1)}\right),$$

which turns out to be a Čech-de Rham cocycle since $\delta\gamma^{(-1,p-1)} = D\tau$. Now, from Weil's theorem (see subsection A.4.3) we conclude that since $\overset{\mathbb{Z}}{\tau}{}^{(p-1,-1)}$ is integral the global form $\gamma^{(-1,p-1)}$ has integral periods. Conversely, if $\gamma^{(-1,p-1)}$ has integral periods then, still from Weil's theorem, it gives rise to an integral Čech-de Rham cocycle $(\tau^{(-1,p-1)} = -\gamma^{(-1,p-1)}, \tau^{(0,p-2)}, \cdots, \tau^{(p-2,0)}, \overset{\mathbb{Z}}{\tau}{}^{(p-1,-1)})$ such that $\delta\gamma^{(-1,p-1)} = D\tau$. This shows that $\mathrm{Ker}(\hat{\phi})$ is nothing else but the space of $(p-1)$-forms with integral periods. So we can extend our exact sequence to the left using the canonical injection of $(p-1)$-forms with integral periods into $(p-1)$-forms

$$\left\{\begin{array}{c}\text{Closed global } (p-1)\text{-forms}\\ \text{with integral periods}\end{array}\right\} \xrightarrow{i} \Omega^{(-1,p-1)}(\mathbb{R}) \xrightarrow{\hat{\phi}} H^p(\mathcal{C}_p, D) \xrightarrow{\hat{\pi}} H^p_{\text{Čech}}(M,\mathbb{Z}) \longrightarrow 0\,.$$

Finally, it is obvious that $\mathrm{Ker}(i) = 0$. This last point definitively establishes the exactness of (III) for $p = q \neq 0$.

In the special case $p = q = 0$, identity $(**)$ reads

$$(1+K\tilde{d})^{-1}\,\overset{\mathbb{Z}}{c}{}^{(0,-1)} + \delta\gamma^{(-1,-1)} = \overset{\mathbb{Z}}{c}{}^{(0,-1)} + \delta\gamma^{(-1,-1)},$$

where $\gamma^{(-1,-1)}$ is just a real number. This means that reduced representatives of $[c] \in H^0(\mathcal{C}_0, D)$ are integral Čech cohomology classes (canonically imbedded in the real Čech cohomology). Conversely, any Čech cohomology class $\xi \in H^0_{\text{Čech}}(M, \mathbb{Z})$ defines a Deligne-Beilinson class $[(1+K\tilde{d})^{-1} \overset{\mathbb{Z}}{c}{}^{(0,-1)}]$, i.e. $H^0(\mathcal{C}_0, D) \simeq H^0_{\text{Čech}}(M, \mathbb{Z})$. As a side note, this can be combined with (II) to yield the more general result

$$H^q(\mathcal{C}_0, D) \simeq H^q_{\text{Čech}}(M, \mathbb{Z}) .$$

If $H^p_{\text{Čech}}(M, \mathbb{Z})$ has no torsion, the sequence (III) is split : choose a basis of $H^p_{\text{Čech}}(M, \mathbb{Z})$, take a representative Čech cocycle in $\Omega^p(M, \mathbb{Z})$ for each basis element, and multiply it by $(1+K\tilde{d})^{-1}$ to get a Deligne-Beilinson cocycle, then extend by linearity. This gives an injection of $H^p_{\text{Čech}}(M, \mathbb{Z})$ into $H^p(\mathcal{C}_p, D)$ which is isomorphic (as an abelian group, but in a non canonical way) to

$$H^p_{\text{Čech}}(M, \mathbb{Z}) \oplus \Omega^{p-1}(M, \mathbb{R}) / \left\{ \begin{array}{c} \text{Closed global } (p-1) \text{ forms} \\ \text{with integral periods} \end{array} \right\} .$$

If $H^p_{\text{Čech}}(M, \mathbb{Z})$ has torsion there is no splitting and the above description is not correct. Finally note the special case $p = q = 1$: $H^1(\mathcal{C}_1, D)$ is canonically isomorphic to $C^\infty(M, \mathbb{R}/\mathbb{Z})$, the multiplicative group of smooth functions from M to the circle group, a more compact description than the one given by the exact sequence (III).

A.7 The isomorphism between Cheeger-Simons differential characters and Deligne-Beilinson classes for $q = p$

The Deligne-Beilinson cohomology group can be imbedded into another exact sequence

$$0 \longrightarrow H^{p-1}_{\text{Čech}}(M, \mathbb{R}/\mathbb{Z}) \longrightarrow H^p(\mathcal{C}_p, D) \longrightarrow \Omega^p_{\mathbb{Z}}(M, \mathbb{R}) \longrightarrow 0 ,$$

which fits better with the representation we have chosen for the classes, namely:

$$\omega = (\omega^{(0,p-1)}, \cdots, \omega^{(p-1,0)}, \overset{\mathbb{Z}}{\omega}{}^{(p,-1)}) .$$

On the other hand, the Cheeger-Simons differential character group $\hat{H}^p(M, \mathbb{R}/\mathbb{Z})$ can also be imbedded into the same exact sequence [2, 14]

$$0 \longrightarrow H^{p-1}_{\text{Čech}}(M, \mathbb{R}/\mathbb{Z}) \longrightarrow \hat{H}^p(M, \mathbb{R}/\mathbb{Z}) \longrightarrow \Omega^p_{\mathbb{Z}}(M, \mathbb{R}) \longrightarrow 0 .$$

These two sequences can be combined into the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{p-1}_{\text{Čech}}(M, \mathbb{R}/\mathbb{Z}) & \longrightarrow & H^p(\mathcal{C}_p, D) & \longrightarrow & \Omega^p_{\mathbb{Z}}(M, \mathbb{R}) & \longrightarrow & 0 \\
 & & \text{id} \updownarrow & & \downarrow \int & & \text{id} \updownarrow & & \\
0 & \longrightarrow & H^{p-1}_{\text{Čech}}(M, \mathbb{R}/\mathbb{Z}) & \longrightarrow & \hat{H}^p(M, \mathbb{R}/\mathbb{Z}) & \longrightarrow & \Omega^p_{\mathbb{Z}}(M, \mathbb{R}) & \longrightarrow & 0
\end{array}$$

in which the descending map in the middle -$\int$- is given by (3.21). Then by the *5-Lemma* this map is an isomorphism.

JHEP08(2005)027

A.8 The Deligne-Beilinson cohomology is the same for all good coverings

A proof is needed only when $q = p$, because in the other cases, we have given canonical isomorphisms with standard Čech cohomology spaces.

If the simple covering $\mathcal{V}$ of M is a refinement of the simple covering $\mathcal{U}$, it is a classical theorem that for Čech cohomology the restriction chain map induces an isomorphism in cohomology. This isomorphism is canonical because restriction is canonical.

We use this as a starting point to prove the corresponding result for Deligne-Beilinson cohomology. To avoid notational ambiguities, we write $\mathcal{C}_p(\mathcal{U})$ (resp. $\mathcal{C}_p(\mathcal{V})$) for the Deligne-Beilinson complex for the covering $\mathcal{U}$ (resp. $\mathcal{V}$).

Restriction gives a chain map from the complex $(\mathcal{C}_p(\mathcal{U}), D)$ to the complex $(\mathcal{C}_p(\mathcal{V}), D)$. So there is a canonical homomorphism

$$H^p(\mathcal{C}(\mathcal{U})_p, D) \stackrel{\text{restriction}}{\longrightarrow} H^p(\mathcal{C}(\mathcal{V})_p, D)\,.$$

We want to show that this homomorphism is one-to-one onto.[18] We start by showing that the homomorphism is one to one. Suppose that an element of $H^p(\mathcal{C}(\mathcal{U})_p, D)$, represented by a certain $c = (c^{(0,p-1)}, \cdots, c^{(p-1,0)}, \overset{\mathbb{Z}}{c}{}^{(p,-1)}) \in \mathcal{C}(\mathcal{U})^p_p$, maps to the trivial element in $H^p(\mathcal{C}(\mathcal{V})_p, D)$. This implies that the restriction of $c^{(p,-1)}$ to the covering $\{\mathcal{V}_\sigma\}_{\sigma\in J}$ is a trivial Čech cocycle, and by the isomorphism theorem for Čech cohomology, $c^{(p,-1)}$ itself is trivial. From the previous section, we know then that c is Deligne-Beilinson cohomologous to some $\delta\gamma^{(p-1)}$ where $\gamma^{(p)}$ is a global de Rham $(p-1)$ form, so we can assume that $c = \delta\gamma^{(p-1)}$ to start with. The condition of triviality is then the same for both coverings, i.e. $\gamma^{(p-1)}$ has to be closed with integral periods. We have proved that in the diagram

$$\begin{tikzcd}
0 \arrow[d] & & \\
H^p(\mathcal{C}(\mathcal{U})_p, D) \arrow[r] \arrow[d, "r"] & H^p_{\text{Čech}}(M, \mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
H^p(\mathcal{C}(\mathcal{V})_p, D) \arrow[r] & H^p_{\text{Čech}}(M, \mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

the first column is exact (i.e. restriction is one to one) and the kernels of the top and bottom rows are canonically isomorphic via restriction.

To prove that the restriction map is onto, we take compatible partitions of unity $\{\vartheta_\alpha\}$ and $\{\varphi_\sigma\}$ for $\mathcal{U}$ and its refinement $\mathcal{V}$. Take a class in $H^p(\mathcal{C}(\mathcal{V})_p, D)$, represented by a cocycle $s = (s^{(0,p-1)}, \cdots, s^{(p-1,0)}, s^{(p,-1)})$ in $\mathcal{C}(\mathcal{V})^p_p$. Then $s^{(p,-1)}$ is a Čech cocycle for $\mathcal{V}$. If $s^{(p-1,-1)}$ is an integral Čech cochain of degree $(p-1)$ for $\mathcal{V}$, $s + Ds^{(p-1,-1)} = (\cdots, s^{(p,-1)} + \delta s^{(p-1,-1)})$ represents the same Deligne-Beilinson class, so by the isomorphism theorem for Čech cohomology, we can assume without loss of generality that $s^{(p,-1)}$ is the restriction of a Čech cocycle $c^{(p,-1)}$ for $\{\mathcal{U}_\alpha\}_{\alpha\in I}$. We have proved in the previous section

[18] The general canonical isomorphism theorem for two (arbitrary) simple coverings is an automatic consequence of the fact that on a compact manifold two simple coverings have a common simple refinement.

that s is Deligne-Beilinson cohomologous to $(1+K\tilde{d})^{-1}s^{(p,-1)}+\delta\gamma^{(p-1)}=(\ldots,s^{(p,-1)})$ for some global de Rham $(p-1)$-form $\gamma^{(p-1)}$ (in this formula, K, $\tilde{d}$ and δ are with respect to the covering $\mathcal{V}$) so we can assume without loss of generality that s is of that form to start with. Then $(1+K\tilde{d})^{-1}c^{(p,-1)}+\delta\gamma^{(p-1)}$ (where now K, $\tilde{d}$ and δ are with respect to the covering $\mathcal{U}$) is a Deligne-Beilinson cocycle, and (as restriction commutes with K, $\tilde{d}$ and δ), $s=r(c)$. So each element of $H^p(\mathcal{C}(\mathcal{V})_p,D)$ has a representative which is the restriction of a Deligne-Beilinson p-cocycle for $\mathcal{U}$: the restriction chain map leads to a surjective map in Deligne-Beilinson cohomology. Putting things together, the proof that restriction induces a canonical bijection from $H^p(\mathcal{C}(\mathcal{U})_p,D)$ to $H^p(\mathcal{C}(\mathcal{V})_p,D)$ is complete.

B. Deligne-Beilinson dual of a cycle

In this section we present a construction of a "cycle map" which associates a Deligne-Beilinson cohomology class to a given cycle. The kind of duality that is implied is not of the "Poincaré" type, but is rather an analog of Pontrjagin duality for Deligne-Beilinson cohomology.

Let z_p be a *singular* or rather a *de Rham* (cf. section 3.4) integral p-cycle of M and $\mathcal{U}$ a simple cover. We perform the following descent[19] using the singular boundary operator, b, and the Čech coboundary operator, δ:

$$(\delta z_p)_{\alpha_0}=z_{p\,|\alpha_0}=b\;c^0_{p+1,\,\alpha_0}\quad\text{in}\quad\mathcal{U}_{\alpha_0}\,. \tag{B.1}$$

Then

$$b\,(c^0_{p+1,\,\alpha_1}-c^0_{p+1,\,\alpha_0})=z_{p\,|\alpha_1}-z_{p\,|\alpha_0}=0\quad\text{in}\quad\mathcal{U}_{\alpha_0\alpha_1}, \tag{B.2}$$

so that

$$\left(\delta\,c^0_{p+1}\right)_{\alpha_0\alpha_1}:=c^0_{p+1,\,\alpha_1}-c^0_{p+1,\,\alpha_0}=b\,c^1_{p+2,\,\alpha_0\alpha_1}\quad\text{in}\quad\mathcal{U}_{\alpha_0\alpha_1}\,. \tag{B.3}$$

This descent goes on at level k (the fact that the covering is simple is crucial):

$$\delta\,c^k_{p+k+1}=b\,c^{k+1}_{p+k+2} \tag{B.4}$$

and stops for $k=n-p-2$

$$\delta\,c^{n-p-2}_{n-1}=b\;c^{n-p-1}_n\,. \tag{B.5}$$

As usual c^{k+1}_{p+k+2} is defined in $\mathcal{U}_{\alpha_0\cdots\alpha_{k+1}}$.

Finally,

$$\delta\,c^{n-p-1}_n=c^{n-p}_n\quad\text{with}\quad b\,c^{n-p}_n=0\,, \tag{B.6}$$

in each $\mathcal{U}_{\alpha_0\cdots\alpha_{n-p}}$. Hence every $c^{n-p}_{n\;,\alpha_0\cdots\alpha_{n-p}}$ is a integral n-cycle in $\mathcal{U}_{\alpha_0\cdots\alpha_{n-p}}$, so that we can write

$$c^{n-p}_{n\;,\alpha_0\cdots\alpha_{n-p}}=\overset{\mathbb{Z}}{\eta}_{z,\,\alpha_0\cdots\alpha_{n-p}}\cdot\mathcal{U}_{\alpha_0\cdots\alpha_{n-p}}\,, \tag{B.7}$$

once $\mathcal{U}_{\alpha_0\cdots\alpha_{n-p}}$ has been identified with a singular n-cycle in a natural way. Furthermore, the $\overset{\mathbb{Z}}{\eta}_{z,\,\alpha_0\cdots\alpha_{n-p}}$'s define a Čech cocycle in an obvious way. In terms of de Rham currents

$$c^k_{p+k+1}\longrightarrow{\eta_z}^{(k,n-p-k-1)}, \tag{B.8}$$

[19] All chains involved below are integral chains.

the above descent equations read

$$\delta\,\eta_z{}^{(k,n-p-k-1)} = d\eta_z{}^{(k+1,n-p-k-2)} \cdots \delta\,\eta_z{}^{(n-p-2,1)} = d\eta_z{}^{(n-p-1,0)}. \tag{B.9}$$

Now

$$\delta\,\eta_z{}^{(n-p-1,0)} = d_{-1}\eta_z{}^{(n-p,-1)},$$

where one can show, using integration of n-forms with compact supports in $\mathcal{U}_{\alpha_0\cdots\alpha_{n-p}}$, that

$$\eta^{(n-p,-1)}_{z,\alpha_0\cdots\alpha_{n-p}} = \overset{\mathbb{Z}}{\eta}_{z,\,\alpha_0\cdots\alpha_{n-p}}\,.$$

Therefore the sequence

$$\eta^{(n-p-1)}_{\mathcal{D}}(z) = (\eta_z{}^{(0,n-p-1)}, \ldots, \eta_z{}^{(n-p-1,0)}, \overset{\mathbb{Z}}{\eta}_z)$$

fulfilling the descent (B.9) is nothing but a Deligne-Beilinson cocycle with *distribution* coefficients.

The singular homology that was used here (in the intersections of the simple covering) is not the usual one (i.e. with compact support), but rather the “infinite” one where chains may have non-compact supports. Accordingly, the corresponding currents do not necessarily have compact support in the intersections either. Moreover, the Čech cocycle $\overset{\mathbb{Z}}{\eta}_z$ is *a priori* non trivial since it is obtained from a Čech-de Rham descent of the *a priori* non trivial integration current of z. In fact, $\overset{\mathbb{Z}}{\eta}_z$ is a Čech representative of the Poincaré dual of z on M.

Let us have a look at the ambiguities of the descent of the p-cycle z which led to $\eta^{(n-p-1)}_{\mathcal{D}}(z)$. At the level of the currents $\eta_z{}^{(n-p-k,k-1)}$, one can check that ambiguities of Deligne-Beilinson type (3.18) are obviously present. However, since our starting point is the integral current of z, we could also have ambiguities on $\eta_z{}^{(0,n-p-1)}$ corresponding to the restriction of a globally defined closed $(n-p-1)$-current, $\delta\eta_z{}^{(-1,n-p-1)}$. But, since all the currents of our descent **must** be integration currents of integral chains, $\delta\eta_z{}^{(-1,n-p-1)}$ must necessarily be the integration current of a $(p+1)$-cycle. Hence, it produces a Deligne-Beilinson ambiguity. The same argument holds at the bottom of the descent, where our integral chains will only produce integral Čech cochain ambiguities, which are also of Deligne-Beilinson type. In other words, the fact we use integral chains to produce a Deligne-Beilinson cocycle provides us with a canonical Deligne-Beilinson class $[\eta^{(n-p-1)}_{\mathcal{D}}(z)]$ associated with z.[20]

C. U(1) connections as Deligne-Beilinson cohomology classes

Let us briefly recall how connections over U(1)-bundles are related to Deligne-Beilinson cohomology classes [13]. Let $P := P(M, \mathrm{U}(1), E, \pi)$ be a principal U(1)-bundle with total space E over M and projection π. For a given **simple** open covering of $M, \mathcal{U}$, P is described by transition functions $g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \mapsto \mathrm{U}(1)$ which satisfy the cocycle condition

$$g_{\alpha_0\alpha_1} g_{\alpha_1\alpha_2} g_{\alpha_2\alpha_0} = 1\,, \tag{C.1}$$

[20]This result can be obtained using *integrally flat* currents defined in [28], see also [14].

in any intersection $\mathcal{U}_{\alpha_0\alpha_1\alpha_2}$, or equivalently

$$\Lambda_{\alpha_0\alpha_1} + \Lambda_{\alpha_1\alpha_2} + \Lambda_{\alpha_2\alpha_0} := n_{\alpha_0\alpha_1\alpha_2} \in \mathbb{Z}\,, \tag{C.2}$$

with

$$g_{\alpha_0\alpha_1} = \exp(2i\pi\Lambda_{\alpha_0\alpha_1})\,. \tag{C.3}$$

Trivially

$$n_{\alpha_0\alpha_1\alpha_2} - n_{\alpha_0\alpha_1\alpha_3} + n_{\alpha_0\alpha_2\alpha_3} - n_{\alpha_1\alpha_2\alpha_3} = 0\,, \tag{C.4}$$

in $\mathcal{U}_{\alpha_0\alpha_1\alpha_2\alpha_3}$, which means that the collection $n^{(2,-1)}$ defined by these integers is an **integral Čech 2-cocycle** on M.

Given a collection of local sections, a connection $\tilde{A}$ on P induces a collection $(A)_\alpha$ of locally defined 1-forms on M which glue together on every $\mathcal{U}_{\alpha_0\alpha_1}$ according to

$$A_{\alpha_1} - A_{\alpha_0} = g^{-1}_{\alpha_0\alpha_1} dg_{\alpha_0\alpha_1} \;=\; (2i\pi)\, d\Lambda_{\alpha_0\alpha_1}\,. \tag{C.5}$$

We then obtain a family

$$(A^{(0,1)}, \Lambda^{(1,0)}, n^{(2,-1)}) \in \check{C}^{(0)}(\mathcal{U}, \Omega^1(M)) \times \check{C}^{(1)}(\mathcal{U}, \Omega^0(M)) \times \check{C}^{(2)}(\mathcal{U}, \mathbb{Z})\,, \tag{C.6}$$

such that

$$\begin{aligned} (\delta A^{(0,1)})_{\alpha_0\alpha_1} &:= A_{\alpha_1} - A_{\alpha_0} = (2i\pi)\, d\Lambda_{\alpha_0\alpha_1}\,, \\ (\delta\Lambda^{(1,0)})_{\alpha_0\alpha_1\alpha_2} &:= \Lambda_{\alpha_0\alpha_1} + \Lambda_{\alpha_1\alpha_2} + \Lambda_{\alpha_2\alpha_0} = d_{-1} n_{\alpha_0\alpha_1\alpha_2} := n_{\alpha_0\alpha_1\alpha_2}\,, \\ (\delta n^{(2,-1)})_{\alpha_0\alpha_1\alpha_2\alpha_3} &:= n_{\alpha_0\alpha_1\alpha_2} - n_{\alpha_0\alpha_1\alpha_3} + n_{\alpha_0\alpha_2\alpha_3} - n_{\alpha_1\alpha_2\alpha_3} = 0\,, \end{aligned} \tag{C.7}$$

in the appropriate intersections. As described in detail above such a sequence makes up a **Deligne-Beilinson cocycle**.

The curvature of $\tilde{A}$ also admits canonical local representatives on M, $F_\alpha := dA_\alpha$, which are globally defined since

$$F_{\alpha_1} - F_{\alpha_0} = d(A_{\alpha_1} - A_{\alpha_0}) = 2i\pi d(d\Lambda_{\alpha_0\alpha_1}) = 0\,, \tag{C.8}$$

Obviously, the existence of F on M is a direct consequence of the existence of $A^{(0,1)}$, and we can formally write "$F = dA^{(0,1)}$".

For a given triple $(\mathcal{U}, P, \tilde{A})$ the Deligne-Beilinson cocycle $(A^{(0,1)}, \Lambda^{(1,0)}, n^{(2,-1)})$ is not unique. More precisely, ambiguities on the local representatives of P and $\tilde{A}$ (that is allowed changes of transition functions and local sections) induce ambiguities on the Deligne-Beilinson cocycle (C.6) of the following form

$$\left(dq^{(0,0)}, \delta q^{(0,0)} + d_{-1} m^{(1,-1)}, \delta m^{(1-,1)}\right), \tag{C.9}$$

with $(m^{(1,-1)}, q^{(0,0)}) \in \check{C}^{(1)}(\mathcal{U}, \mathbb{Z}) \times \check{C}^{(0)}(\mathcal{U}, \Omega^0(M))$. Such ambiguities correspond precisely to Deligne-Beilinson coboundaries and thus represent the ambiguities among the representatives of the relevant Deligne-Beilinson cohomology classes.[21]

[21] See appendix A

Two triples $(\mathcal{U}, P, \tilde{A})$ and $(\mathcal{U}, P', \tilde{A}')$ are said to be U(1)-*equivalent* if there is a U(1) isomorphism $\Phi : P \mapsto P'$, such that $\tilde{A}' = \Phi_* \tilde{A}$. Locally, this means that the transition functions of P and P' are related according to

$$g'_{\alpha\beta} = h_\alpha{}^{-1} \cdot g_{\alpha\beta} \cdot h_\beta \, , \tag{C.10}$$

or equivalently

$$\Lambda'_{\alpha\beta} = \Lambda_{\alpha\beta} + q_\beta - q_\alpha \, , \tag{C.11}$$

where the $h_\alpha = \exp(2i\pi q_\alpha)$. In the same way the local representatives of the connections fulfill

$$A'_\alpha = A_\alpha + 2i\pi dq_\alpha \, . \tag{C.12}$$

Then we clearly see that these relations assume the same form as the ambiguities in (C.9), showing that two equivalent triples are associated to the same Deligne-Beilinson cohomology class in $H^2_{\mathcal{D}}(M, \mathbb{Z}(2))$.

This correspondence can be established in the reverse way. Indeed, consider a representative $(A^{(0,1)}, \Lambda^{(1,0)}, n^{(2,-1)})$ of a Deligne-Beilinson cohomology class, the U(1)-valued mappings $g_{\alpha\beta} := \exp 2i\pi\Lambda_{\alpha\beta}$ are U(1) transition functions over $\mathcal{U}$ since they satisfy the cocycle condition (C.1). With these functions, one can canonically build a principal U(1)-bundle over M, $P(M, \mathrm{U}(1), E, \pi)$ [29, 30]. Furthermore, there is only one connection $\tilde{A}$ on P whose local representatives on M coincide with those of $A^{(0,1)}$. Hence our Deligne-Beilinson cocycle defines a couple $(P, \tilde{A})$ in a canonical way.

Now, with another representative, $(A^{(0,1)} + dq^{(0,0)}, \Lambda^{(1,0)} + \delta q^{(0,0)}, n^{(2,-1)})$, we obtain another set of transition functions which defines an equivalent principal bundle — cf. (C.10). In the same way, $A^{(0,1)} + dq^{(0,0)}$ is related to $\tilde{A}$ through a U(1)-bundle isomorphism.

Finally, a representative $(A^{(0,1)}, \Lambda^{(1,0)} + m^{(1,-1)}, n^{(2,-1)} + \delta m^{(1,-1)})$ gives the same transition functions and the same connection. This establishes that the Deligne-Beilinson cohomology class of $(A^{(0,1)}, \Lambda^{(1,0)}, n^{(2,-1)})$ can be canonically associated to a whole class of U(1)-equivalent triples $(\mathcal{U}, P, \tilde{A})$.

The independence of this isomorphism upon the chosen covering $\mathcal{U}$ of M is a direct consequence of the results proven in (A.8).

Note added. While completing this paper, we became aware of the recent mathematical work of R. Harvey, B. Lawson and J. Zweck [14], who discuss in detail the Pontrjagin duality we use in section 3.3. The authors emphasize the differential character point of view rather than the Deligne-Beilinson one we have adopted here.

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